Three-tier CFTs from Frobenius algebras

A. Henriques
Utrecht University, the Netherlands

Notes compiled by J. Lamers

Abstract

These are lecture notes of a course given at the Summer School on Topology and Field Theories held at the Centre for Mathematics of the University of Notre Dame, Indiana, from May 29 to June 2, 2012.

The idea of extending quantum field theories to manifolds of lower dimension was first proposed by Dan Freed in the nineties. In the case of conformal field theory (CFT), we are talking of an extension of the Atiyah-Segal axioms, where one replaces the bordism category of Riemann surfaces by a suitable bordism bicategory, whose objects are points, whose morphism are 1-manifolds, and whose 2-morphisms are pieces of Riemann surface.

There is a beautiful classification of full (rational) CFT due to Fuchs, Runkel and Schweigert, which roughly says the following. Fix a chiral algebra A (= vertex algebra). Then the set of full CFTs whose left and right chiral algebras agree with A is classified by Frobenius algebras internal to $\mathrm{Rep}\,(A)$. A famous example to which one can successfully apply this is the case where the chiral algebra A is affine $\mathfrak{su}(2)$ at level k, for some $k \in \mathbb{N}$. In that case, the Frobenius algebras in $\mathrm{Rep}\,(A)$ are classified by A_n, D_n, E_6, E_7, E_8, and so are the corresponding CFTs.

Recently, Kapustin and Saulina gave a conceptual interpretation of the FRS classification in terms of 3-dimensional Chern-Simons theory with defects. Those defects are also given by Frobenius algebra object in $\mathrm{Rep}\,(A)$. Inspired by the proposal of Kapustin and Saulina, we will (partially) construct the three-tier CFT associated to a Frobenius algebra object.

Contents

1 Introduction

In these notes we define, and partially construct, extended conformal field theories starting from a so-called chiral conformal field theory, and a Frobenius algebra object.

The idea of *extended* field theory, which goes back to the work of Freed in the ninetees [Fre93], started in the context of topological field theory. There, it is an extension of Atiyah's definition of topological quantum field theory (TQFT) [Ati89] where, instead of just assigning vector spaces to $(d-1)$-dimensional manifolds and linear maps to d-dimensional cobordisms, one also assigns data to manifolds of lower dimension, all the way down to points. Thus, the extended field theory consists of $d+1$ tiers.

Extended *conformal* field theories (CFTs) were first proposed by Stolz and Teichner [ST04], in the context of their project of constructing elliptic cohomology, and then also mentioned in a review paper by Segal [Seg07]. However, they did not provide any constructions of extended CFTs. We will show that this can be done, at least to a great extent.

1.1 Outline

Let us briefly outline the content of these notes. In Section 2 we introduce (full) CFT[1] in the formalism of Graeme Segal, and define extended CFT. The source and target bicategories of extended CFT are discussed in some detail.

Section 3 contains a discussion of chiral CFTs. We introduce the two important ingredients of our construction: conformal nets, and Frobenius algebra objects. We also recall some aspects of the construction of Fuchs, Runkel and Schweigert, which constructs a (non-extended) full CFT from a chiral CFT and a Frobenius algebra object in the associated category.

In Section 4 we describe work in progress: the construction of an extended CFT from a conformal net, and a Frobenius algebra object in the representation category of the conformal net. We finish by describing the main unsolved problem, namely the construction of the bimodule map that corresponds to a surface with four cusps (the 'ninja star' in Figure 1). If this could be done, this would complete the construction of the full CFT.

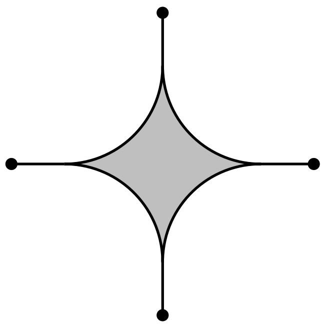


Figure 1: A 'ninja star' is a 2-surface with four cusps. Our main open problem is to construct the corresponding map of bimodules.

2 Extended conformal field theory

The definition of extended CFT is an extension of Segal's definition of CFT. We start at the beginning, and introduce CFT in Segal's formalism. We will also discuss the notion of conformal welding, which is a necessary ingredient of the definition.

2.1 Segal's definition of conformal field theory

There are several (non-equivalent) ways to define conformal field theory. Although Segal's definition [Seg88, Seg04] is not the most mainstream one, it is the one that has become popular amongst mathematicians.

Definition (Segal)**.** A full[2] *conformal field theory* is a symmetric monoidal

[1]In this paper, "CFT" will always refer to two-dimensional conformal field theory.

[2] There also exists another notion, called chiral CFT. This will be discussed in Section 3. Until then, all CFTs will be full CFTs.

functor from the category of conformal cobordisms, which consists of

$$\begin{cases} \text{objects: one-dimensional, compact, oriented, smooth manifolds;} \\ \text{morphisms: cobordisms equipped with a complex structure;} \\ \text{monoidal structure: taking disjoint unions;} \end{cases}$$

to the category of Hilbert spaces, with

$$\begin{cases} \text{objects: Hilbert spaces;} \\ \text{morphisms: bounded linear maps;} \\ \text{monoidal structure: usual tensor product of Hilbert spaces.} \end{cases}$$

Let us take a closer look at the category of conformal cobordisms. Its objects consist of (possibly empty) disjoint unions of oriented circles, always with a smooth structure. A CFT maps such a disjoint unions of circles to some Hilbert space, referred to as the 'state space' by physicists. The empty manifold is the unit object for the monoidal structure, and is sent to the trivial Hilbert space $\mathbb{C}$, which is the unit for the tensor product in the category of Hilbert spaces.

The morphisms are Riemann surfaces with boundary. Both the smooth structure and the complex structure extend all the way to the boundary of the cobordisms. Alternatively, one could take the complex structure to only be defined on the interior, and require that the cobordism be locally isomorphic to the upper half plane. The orientations of the one-manifolds have to be compatible with those of the cobordisms connecting them: if Σ is a cobordism from S to S', then by definition there exists an orientation preserving diffeomorphism from the boundary $\partial\Sigma$ of Σ to the disjoint union $S \amalg \overline{S'}$ of the 'ingoing' manifold S and the 'outgoing' S' with orientation reversed.

A CFT sends cobordisms to linear maps between Hilbert spaces, the 'propagator' or 'correlator'. In particular, a closed cobordism Σ between two empty manifolds is mapped to a linear map $\mathbb{C} \longrightarrow \mathbb{C}$. The latter is completely determined by a single complex number $Z(\Sigma)$, the 'partition function' at the Riemann surface. Although the category of conformal cobordisms does not come with identity morphisms, this does not pose problems for the definition of a CFT. Alternatively, one can add identities by hand and think of them as infinitesimally thin cobordisms. More interesting is to also allow for cobordisms that are partially thin, and partially thick, such as the one in Figure 2.

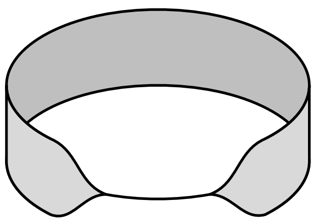


Figure 2: A partially thin annulus.

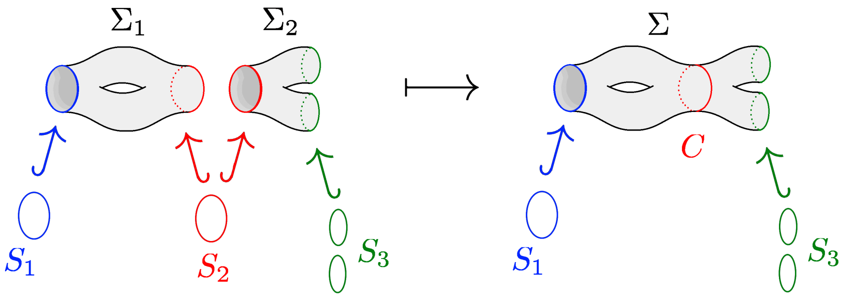


Figure 3: Conformal welding of two composable cobordisms.

2.1.1 Conformal welding

The composition of conformal cobordisms is tricky and deserves special attention. We outline the procedure, which is called *conformal welding*. Consider two composable cobordisms Σ_1 and Σ_2 as depicted in Figure 3. As topological spaces, Σ_1 and Σ_2 can be glued in the obvious way. However, a priori, the composition Σ is only equipped with a smooth structure *away* from the curve C along which Σ_1 and Σ_2 have been glued, and similarly for the complex structure. These issues are resolved by the following theorem [Seg, RS06].

Theorem 2.1. *In the above situation, there exists a unique complex structure on the interior of the topological manifold Σ which is compatible with the given complex structures on Σ_1 and Σ_2. Moreover, the embedding of C into Σ is smooth.*

Note that the embedding $C \hookrightarrow \Sigma$ will typically not be analytic; this already signals that the proof of the theorem will have to be rather involved. A closely related result, which is needed in the proof that conformal welding is well defined, is [Bel90]:

Lemma 2.2. *Let $D \subset \mathbb{C}$ be a connected, simply connected open subset of the complex plane, and let us assume that the boundary of D is smooth. Let $D_0 \subset \mathbb{C}$ be the standard disc centered at the origin. Then the map $D \longrightarrow D_0$ provided by the Riemann mapping theorem is smooth all the way to the boundary.*

Since the problem in Theorem 2.1 is local, one can reduce the general problem of conformal welding to the simpler situation of glueing two discs along a smooth identification of their boundaries. Moreover, using Lemma 2.2, one can further reduce the problem to that of glueing two standard discs along a smooth identification of their boundaries. Theorem 2.1 is therefore equivalent to the following special case of the theorem: given two standard discs D_0 and D_0' in $\mathbb{C}$, and a diffeomorphism φ between their boundaries, the resulting glued surface is a copy of the Riemann sphere $\mathbb{CP}^1$, along with a smoothly embedded curve in it, as shown in Figure 4.

In order to get an extended CFT, both the source and target categories in Segal's definition of a CFT are replaced by appropriate bicategories. An extended CFT is then simply a symmetric monoidal functor between these bicategories. We first discuss the source bicategory.

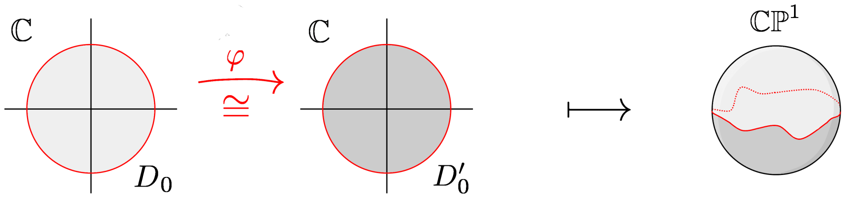
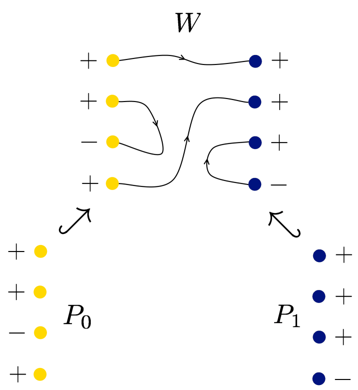


Figure 4: Glueing two standard discs along their boundaries results in the Riemann sphere with a smoothly embedded curve.

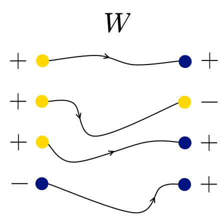
2.2 The source bicategory: conformal surfaces with cusps

Geometrically, an *extended cobordism* is a cobordism, say d-dimensional, whose boundary comes in two pieces where each piece is viewed as $(d-1)$-dimensional cobordisms, and so on. In the case of CFTs one is interested in $d = 2$, resulting in three tiers: zero-manifolds, one-manifolds, and two-manifolds.

Before describing the source category in more detail, let us give the geometrical picture. Starting in dimension zero, we first have zero-dimensional, oriented manifolds: these are disjoint unions of points, each of which is labelled $+$ or $-$ indicating the orientation. Moving up one dimension, we have cobordisms between the zero-dimensional manifolds.



(a) An example of a one-dimensional cobordism $W\colon P_0 \longrightarrow P_1$.



(b) The same cobordism, depicted in a more convenient way.

Figure 5: The figure on the left shows a one-dimensional cobordism. Notice that the orientation of the points is related to the orientation of the cobordisms connecting them, as required by the definition of a cobordism. The figure on the right shows the same cobordism displayed in way that will be more suitable for illustrations of two-dimensional cobordisms. The incoming zero-manifold is drawn in yellow, and the outgoing zero-manifold in dark blue. The inclusion of the boundary manifolds is understood.

An example of such a one-dimensional cobordism shown in Figure 5a, all 'incoming' zero-manifolds are on the left, and all 'outgoing' on the right. To facilitate drawing the more complicated examples below it is convenient to employ a different convention, and use colors to represent whether a zero-manifold

is incoming or outgoing, respectively. With this convention, the example from Figure 5a can also be represented as in Figure 5b.

Circles, which form the objects of the source category of a non-extended CFT, fit in the formalism of extended CFT as closed cobordisms between empty zero-manifolds. They can be obtained from intervals by glueing. Going up one more dimension, cobordisms between such closed cobordisms are the conformal cobordisms that we encountered before: these are Riemann surfaces with boundary. However, now we also have two-dimensional cobordisms with cusps such as the examples in Figure 6.

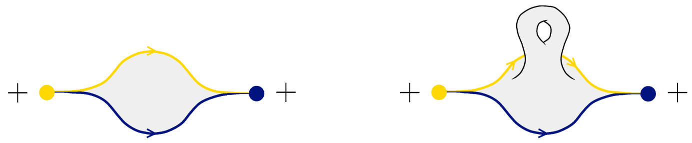


Figure 6: Two examples of two-dimensional cobordisms with cusps. The cobordism on the right has a nontrivial topology. Like in Figure 5b, the colors indicate which one-dimensional boundary is incoming and which is outgoing.

The source category is the bicategory of conformal surfaces with cusps, which is defined as follows. It has

- objects: zero-dimensional, oriented manifolds P.
- 1-morphisms: one-dimensional cobordisms $P_0 \hookrightarrow W \hookleftarrow P_1$ with smooth structure, and collars $P_0 \times [\,0, \varepsilon) \longrightarrow W$ and $P_1 \times (-\varepsilon, 0\,] \longrightarrow W$ parametrizing the ends.
- 2-morphisms: two-dimensional cobordisms $W_0 \hookrightarrow \Sigma \hookleftarrow W_1$ of cobordisms, with conformal structure in the interior $\Sigma \setminus (W_0 \cup W_1)$, and such that the diagrams

$$\begin{tikzcd} P_0 \times [\,0,\varepsilon) \arrow[r] \arrow[d] & W_0 \arrow[d] \\ W_1 \arrow[r] & \Sigma \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P_1 \times (-\varepsilon,0\,] \arrow[r] \arrow[d] & W_0 \arrow[d] \\ W_1 \arrow[r] & \Sigma \end{tikzcd}$$

  commute, after maybe shrinking ε. Furthermore, Σ should be locally isomorphic to one of the local models specified in Section 2.2.1 below.

The two diagrams above say that the parametrizations of the one-dimensional cobordisms bounding the surface have to agree on neighbourhoods of their ends. In particular, this forces the two-dimensional cobordism Σ to be in fact one-dimensional near the zero-manifolds P_0 and P_1. Figure 7 shows an example of a 2-morphism in the category of conformal surfaces with cusps. Taking disjoint unions endows the category of conformal surfaces with cusps with a symmetric monoidal structure.

2.2.1 Local models

The various manifolds comprising the bicategory of conformal surfaces admit local models. Being a local model means that any point of such a manifold has

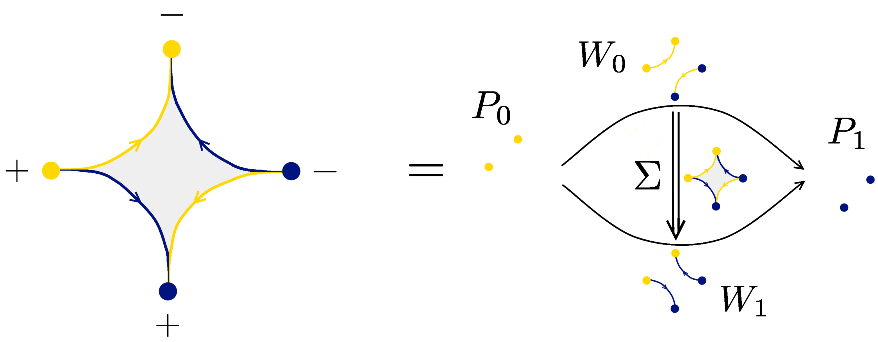


Figure 7: An example of a conformal surface with cusps, and the corresponding 2-morphism $\Sigma\colon W_0 \longrightarrow W_1$.

a neighbourhood that looks the same as some open subset of the corresponding local model. For example, the local model of an object P is simply a point with a choice of orientation, and for a 1-morphism S it is the unit interval $[0,1]$. Unlike for the case of 0-morphisms (objects) and 1-morphisms, where one can show that, locally, they must look like one of the local models, the case of 2-morphisms is different. For 2-morphisms, giving the list of allowed local models is part of the definition of what things we allow as 2-morphisms.

We can describe the local models for our 2-morphisms Σ as follows. Let $f, g \in C^\infty\big([0,1],\mathbb{R}\big)$ be smooth functions on the unit interval, such that $f \le g$, and such that f and g are equal on neighbourhoods of 0 and 1. Then a local model of Σ is

$$\Sigma = \{\, x + i\,y \mid f(x) \le y \le g(x) \,\} \ . \tag{1}$$

In particular, since we require f and g to agree near the ends, the tips of Σ are really one-dimensional cusps as depicted in Figure 8. There are many different local models for the 2-morphisms, with different choices for f and g yielding varying degrees of 'sharpness' for the cusps.

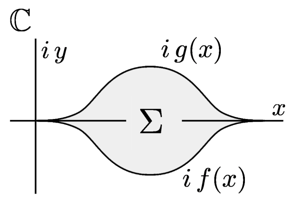


Figure 8: A local model as described by equation (1). Different choices for $f, g \in C^\infty\big([0,1],\mathbb{R}\big)$ agreeing near the endpoints give rise to different degrees of sharpness for the cusps.

Since the 1-morphisms correspond to collared one-manifolds, they can be composed by glueing. To see that the glued surfaces with cusps are again of the prescribed form, notice that our problem is local. Thus we may assume without loss of generality that the surfaces we want to glue are given by the

local models. It is clear from Figure 9 that the horizontal composition is again of the form (1).

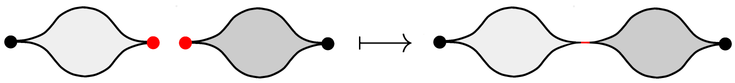


Figure 9: Horizontal composition of two-dimensional local models.

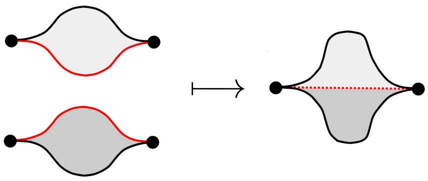


Figure 10: Vertical composition of two local models for conformal surfaces with cusps. It requires some work to show that the result is a local model too.

The vertical composition of two 2-morphisms looks as shown in Figure 10. We can use Theorem 2.1, underlying conformal welding, to get a complex structure on the interior of the glued surface. But a priori, it is not clear that the result is again one of our local models. To show that, we will use Lemma 2.2. First we get rid of the corners by embedding the two surfaces that we want to glue into discs with a smooth boundary, as shown in Figure 11. Next, we extend the diffeomorphism of the boundaries of the surfaces that we want to identify to a diffeomorphism between the boundary circles, and glue. The result is depicted in Figure 12. By Lemma 2.2, everything is smoothly embedded in $\mathbb{CP}^1$. This shows that the glued surface is again one of our allowed local models, and so it is again a 2-morphism.

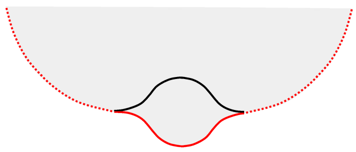


Figure 11: A local model Σ can be embedded into some disc whith smooth boundary in the complex plane.

2.3 The target bicategory: von Neumann algebras

Since an extended CFT should encompass the notion of CFT, it should certainly map (a union of) circles to some Hilbert space, and a cobordism connecting such circles to a linear map between Hilbert spaces, as before. We have to decide what we want to assign to a point: these should be some kind of algebras. If we want to stay in a Hilbert space setting, then there are not many options for the kind of algebras to consider. It turns out that the appropriate choice is given

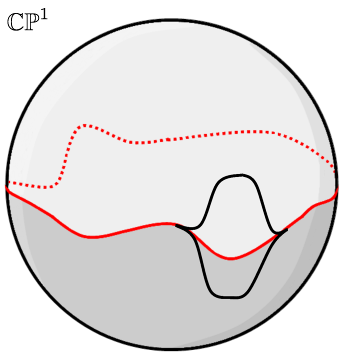


Figure 12: Glueing two local models Σ_1 and Σ_2 that are embedded into discs like in Figure 11 yields a copy of the vertical composition of Σ_1 and Σ_2 inside the Riemann sphere.

by von Neumann algebras. A one-dimensional cobordism is then mapped to a bimodule between von Neumann algebras, and a surface such as (1) corresponds to a linear map between bimodules. In short, the target bicategory is defined as follows:

- objects: von Neumann algebras;
- 1-morphisms: bimodules (that is, Hilbert spaces with a left action of the first von Neumann algebra, and a commuting right action of the second von Neumann algebra);
- 2-morphisms: bounded linear maps that are compatible with the bimodule structure.

Before we give a definition of these notions, we recollect some facts from the theory of operator algebras. Given a Hilbert space H, denote the algebra of bounded operators on H by $B(H)$. Recall that an operator $a \in B(H)$ is *trace class* if it is compact and the trace-norm $\|a\|_1 := \sum_k \sqrt{\mu_k}$ is finite; here the μ_k are the eigenvalues of the positive operator a^*a. This ensures that the trace of a is well defined. Write $B_1(H)$ for the trace-class operators in $B(H)$. The pairing

$$B(H) \times B_1(H) \longrightarrow \mathbb{C} \ , \quad (a,b) \longmapsto \mathrm{tr}\,(a\,b)$$

induces a topology on $B(H)$ that is called the *ultraweak* topology. Thus, a (generalized) sequence $\{a_i\}$ in $B(H)$ converges ultraweakly to $a \in B(H)$ if and only if for all $b \in B_1(H)$ we have that $\mathrm{tr}(a_i b) \longrightarrow \mathrm{tr}(ab)$ in $\mathbb{C}$.

Definition. A *von Neumann algebra* is a topological $*$-algebra[3] A over $\mathbb{C}$ that can be embedded in some $B(H)$ as a ultraweakly closed $*$-subalgebra.

By the von Neumann bicommutant theorem, A is ultraweakly closed if and only if it is its own bicommutant.

[3]Notice that the multiplication map $B(H) \times B(H) \longrightarrow B(H)$ is not continuous, so the term "topological $*$-algebra" should be taken with a grain of salt.

Definitions. A *module* over a von Neumann algebra A is a Hilbert space H together with a continuous $*$-homomorphism $A \longrightarrow B(H)$.

Similarly, if A and B are von Neumann algebras, an *A-B-bimodule* is a Hilbert space H equipped with two continuous $*$-homomorphisms $A \longrightarrow B(H)$ and $B^{\mathrm{op}} \longrightarrow B(H)$ whose images commute. We write ${}_AM_B$ to indicate that M is an A-B-bimodule.

Here we have written B^{op} for the von Neumann algebra obtained from B by reversing the order in the multiplication: if $m\colon B \otimes B \longrightarrow B$ is the original multiplication on B then the opposite multiplication is given by $m^{\mathrm{op}}(a,b) = b\,a$.

It is more work to define the composition of bimodules in the bicategory of von Neumann algebras than it is to do so in the bicategory of rings. Recall the way in which rings and bimodules form a bicategory. Given two rings R and S, let $\mathrm{Hom}(R,S)$ be the category of R-S-bimodules. The morphisms in $\mathrm{Hom}(R,S)$ are then the 2-morphisms of our bicategory. If R, S, T are rings, the horizontal composition of two bimodules ${}_RM_S$ and ${}_SN_T$ is given by the tensor product:

$$ {}_AM_B \circ {}_BN_C \coloneqq {}_AM \underset{B}{\otimes} N_C \ . $$

Here the tensor product is taken over B, so that $(m \cdot b) \otimes n = m \otimes (b \cdot n)$. The A-A-bimodule A is then the unit object for this kind of composition.

If we want to do something similar with von Neumann algebras, the first obstacle is the definition of the unit object: a von Neumann algebra is not a Hilbert space, so it cannot serve as a bimodule over itself. However, there is a canonical way to turn a von Neumann algebra A into a Hilbert space, called L^2A.

2.3.1 The L^2-space of a von Neumann algebra

The definition of L^2A requires more prerequisites from the theory of operator algebras. We outline its construction.

For A a von Neumann algebra, let

$$ \begin{aligned} L^1A &\coloneqq \{\ \varphi\colon A \longrightarrow \mathbb{C} \mid \text{continuous}\ \}\ , \\ L^1_+A &\coloneqq \{\ \varphi \in L^1A \mid \varphi(a^*a) \geq 0 \text{ for all } a \in A\ \}\ . \end{aligned} $$

Elements of L^1_+A are called *states*[4] on A. The GNS-construction says that for each state $\varphi \in L^1_+A$ there exists a cyclic representation π_φ of A on some Hilbert space H_φ with cyclic vector Ω_φ. Thus, the image $\pi_\varphi(A)\,\Omega_\varphi$ of the action of A on Ω_φ is dense in H_φ.

If the state is faithful (i.e. if $\varphi(a^*a) > 0$ for $a \neq 0$), then the antilinear operator $\pi_\varphi(a)\,\Omega_\varphi \longmapsto \pi_\varphi(a)^*\,\Omega_\varphi$ defined on $\pi_\varphi(A)\,\Omega_\varphi$ can be extended to an operator S_φ on the closure H_φ of $\pi_\varphi(A)\,\Omega_\varphi$. From this operator we can further construct the positive operator $\Delta_\varphi \coloneqq |S_\varphi|^2 = S_\varphi^* S_\varphi$. Since Δ_φ is a positive operator, $\Delta_\varphi^{it} = \exp(i\,t\log\Delta_\varphi)$ is well defined for all $t \in \mathbb{R}$.

By a theorem that is due to Tomita and Takesaki, for each $a \in A$, the assignment $t \longmapsto \Delta_\varphi^{-it}\,a\,\Delta_\varphi^{it}$ defines a one-parameter family of elements in A. This is called the *modular group* of A associated with φ.

[4] Often, one also puts the condition that $\varphi(1) = 1$.

Next, consider the algebra $\mathrm{Mat}_2(A)$ of 2×2 matrices with coefficients in A and let $\varphi \oplus \psi \in L^1_+\big(\mathrm{Mat}_2(A)\big)$. Via the above construction, $\varphi \oplus \psi$ yields a modular group in $\mathrm{Mat}_2(A)$. Applying this modular group to the element

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(A)$$

we get elements that are of the form

$$\Delta^{-it}_{\varphi\oplus\psi} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta^{it}_{\varphi\oplus\psi} = \begin{pmatrix} 0 & \cdots \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(A)\,.$$

The *non-commutative Radon-Nikodym derivative* $[D\varphi : D\psi]_t$ is the defined via

$$\begin{pmatrix} 0 & [D\varphi : D\psi]_t \\ 0 & 0 \end{pmatrix} := \Delta^{-it}_{\varphi\oplus\psi} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta^{it}_{\varphi\oplus\psi}\ .$$

Now consider the free vector space on symbols $\sqrt{\varphi}$ with $\varphi \in L^1_+ A$. The above construction allows us to define a (semi-definite) inner product on this vector space via the formula

$$\langle \sqrt{\varphi}, \sqrt{\psi}\, \rangle := \underset{t \longrightarrow i/2}{\text{anal. cont.}}\ \varphi\big([D\varphi : D\psi]_t\big)\ .$$

After all these preliminaries, we are finally in a position to define L^2A: it is the Hilbert space obtained as the completion of the above free vector space with respect to this inner product. For each von Neumann algebra A, the Hilbert space L^2A is an A-A-bimodule, ${}_AL^2A_A$, and this is the unit morphism in the bicategory of von Neumann algebras.

The Hilbert space L^2A is also equipped with a *positive cone* $L^2_+A \subset L^2A$, given by

$$L^2_+A := \{\sqrt{\varphi}\,|\,\varphi \in L^1_+A\}\,,$$

and an antilinear involution $J : L^2A \to L^2A$, called the *modular conjugation.* The modular conjugation is given by $J(\sum_i c_i\sqrt{\varphi_i}) := \sum_i \bar{c}_i\sqrt{\varphi_i}$.

2.3.2 Connes fusion

The second difficulty towards defining the bicategory of von Neumann algebras is that the ordinary tensor product does not work: it would have ${}_AA_A$ as its unit, not ${}_AL^2(A)_A$. The appropriate tensor product of von Neumann bimodules, known as *Connes fusion* and denoted by $\boxtimes$, is tailor-made so that ${}_AL^2A_A$ is a unit for that operation. We have

$$M \underset{A}{\boxtimes} N := \text{completion of } M \underset{A}{\otimes} \mathrm{Hom}_A(L^2A, N)\,. \tag{2}$$

This is actually forced on us if we want L^2A to be the unit. If we accept for a moment that L^2A is a unit, then given an A-linear map $\varphi\colon L^2A \longrightarrow N$ and an element $m \in M$, there is an easy way of producing an element of $M \boxtimes_A N$: take the image of $m \in M \cong M \boxtimes_A L^2A$ under the map $1 \boxtimes \varphi : M \boxtimes_A L^2A \to M \boxtimes_A L^2N$.

The completion is taken with respect to an inner product on the right-hand side of (2). Let us work backwards to figure out the correct formula for the inner product. The inner product of two elements $n \otimes \phi$ and $m \otimes \psi$ of $M \boxtimes_A N$ can be described as the composition

$$\begin{tikzcd} \mathbb{C} \arrow[r, "n"] & M \arrow[r, "\mathrm{id} \boxtimes \phi"] \arrow[rr, bend left, dashed, "\mathrm{id} \boxtimes (\psi^* \circ \phi)"] & M \boxtimes_A N \arrow[r, "\mathrm{id} \boxtimes \psi^*"] & M \arrow[r, "m^*"] & \mathbb{C} \end{tikzcd}$$

where $\phi, \psi \in \mathrm{Hom}_A(L^2A, N)$, ψ^* is the adjoint of ψ, and we view $m, n \in M$ as maps $\mathbb{C} \longrightarrow M$. Notice that the map $\psi^* \circ \phi\colon {}_AL^2A \longrightarrow {}_AL^2A$ commutes with the left action of A on L^2A. Now, one of the properties of L^2A is that endomorphisms of L^2A which are equivariant for the left A-action are given by right multiplication ρ_a by some $a \in A$. Therefore we have that $\psi^* \circ \phi = \rho_a$ for some $a = a_{\psi^* \circ \phi} \in A$. The inner product $\langle m \otimes \phi, n \otimes \psi \rangle$ on $M \otimes \mathrm{Hom}_A(L^2A, M)$ is now given by the composition

$$\mathbb{C} \xrightarrow{\ n\ } M \xrightarrow{\quad \rho_a \quad} M \xrightarrow{\ m^*\ } \mathbb{C}\ .$$

A more symmetric way to write the Connes fusion product is

$$M \underset{A}{\boxtimes} N \cong \mathrm{Hom}_A(L^2A, M) \underset{A}{\otimes} L^2A \underset{A}{\otimes} \mathrm{Hom}_A(L^2A, N)\,.$$

The evaluation map $\mathrm{Hom}_A(L^2A, M) \otimes_A L^2A \longrightarrow M$ relates this description to the previous asymmetric definition: after completion those two descriptions become isomorphic to each other.

3 Conformal nets and Frobenius algebra objects

Before we push on, let us pause a moment to sketch the big picture. Actually, there are a couple of different things called 'CFT'; in particular, physics distinguishes between *chiral* and *full* CFT. It is quite common to use 'CFT' to refer to one of these things, but it may not always be clear from the context to which one. What we have been calling CFT above are really *full* CFTs. We abbreviate 'chiral CFT' to 'χCFT' so that we can continue to use 'CFT' without further specification exclusively for 'full CFT'.

Chiral CFTs can be seen as an intermediate step towards full CFT. The distinction between chiral and full CFT has its origin in physics. Now comes the mathematics to make things more complicated: there exist different mathematical formalisms to talk about χCFT, and to talk about full CFT. We have already discussed Segal's formalism for full CFT. Chiral CFT can be described in the formalism of Segal as well, and there are also approaches using *vertex operator algebras* or *conformal nets*. Shortly we will present χCFT in Segal's formalism, and in Sections 3.2.1 and 3.3 we will also look at the approach via conformal nets.

Recall that the *loop group* of a compact Lie group G is defined as the group of maps from the unit circle into G:

$$L\,G \coloneqq \mathrm{Map}_{C^\infty}(S^1, G)\ . \qquad (3)$$

Loop groups are relevant for us because they lead to vertex operator algebras or conformal nets, and so they provide examples of χCFTs. In order to construct a full CFT out of a χCFT, one needs additional data: a *Frobenius algebra object* in the tensor category associated to the χCFT. In Section 3.4 we will define Frobenius algebra objects, and in Section 3.5 we will illustrate how such an object helps to construct a full CFT out of a χCFT.

To summarize, the situation can be represented as follows:

$$\text{loop groups} \xrightarrow[]{\substack{\text{provide} \\ \text{examples of}}} \text{chiral CFT} \xrightarrow[]{\substack{\text{+ Frobenius} \\ \text{algebra object}}} \text{full CFT}$$

We will show that with the same input, a chiral CFT and a Frobenius algebra object, one can actually do better and construct an *extended* CFT:

$$\begin{tikzcd}
 & \text{extended CFT} \arrow[d, "\text{forget}"] \\
\text{chiral CFT} \arrow[ur, dotted, bend left] \arrow[r, "\substack{\text{+ Frobenius} \\ \text{algebra object}}"] & \text{full CFT}
\end{tikzcd} \tag{4}$$

In Section 4 we will (partially) construct examples of extended CFTs. We should put this task into perspective: already for (non-extended) Segal CFTs, the interesting examples — most notably those coming from loop groups — have *not* completely been constructed. In the spirit of the cobordism hypothesis [Lur09], one could even hope that it is easier to construct extended CFTs than full CFTs.

3.1 Chiral conformal field theory

In this section, we use Segal's formalism to describe (non-extended) χCFT. Recall from Section 2.1 that a non-extended CFT assigns Hilbert spaces to one-dimensional manifolds, and maps between Hilbert spaces to conformal cobordisms. Chiral CFTs have the same source category as full CFTs, but there is an intermediate layer on the side of the target.

A χCFT first assigns to every one-dimensional manifold a category $\mathcal{C}$. To each object $\lambda \in \mathcal{C}$, it further assigns a Hilbert space H_λ. Likewise, a cobordism is mapped to a functor $f\colon \mathcal{C}_{\text{in}} \longrightarrow \mathcal{C}_{\text{out}}$, and for each $\lambda \in \mathcal{C}_{\text{in}}$ we further get a map $H_\lambda \longrightarrow H_{f(\lambda)}$. This is the data of a χCFT. In addition, a χCFT must satisfy certain axioms. Most importantly, the map $H_\lambda \longrightarrow H_{f(\lambda)}$ must depend on the complex structure of the cobordism in a holomorphic fashion. If we fix two one-manifolds W_{in} and W_{out}, then the moduli space of Riemann surfaces with these boundaries is an infinite-dimensional manifold, and has its own complex structure: the functions from this space, mapping points to operators, are required to be holomorphic.

To get a feeling about what this all means, we look at the examples of χCFTs which are provided by loop groups. Let G be a Lie group. To a one-manifold W in the source category we assign the category $\mathcal{C}$ of representations of $\mathrm{Map}_{C^\infty}(W, G)$ (compare with (3)). Each object λ in that category has an underlying Hilbert space, which we denote by H_λ.

We should point out that χCFT in the above formalism are difficult to construct, and despite a lot of hard work (mostly by Huang [Hua97] and Zhu [Zhu96], see also Posthuma [Pos03]) the χCFTs corresponding to loop groups have been constructed to a great extent, but not completely.

3.2 Conformal nets

Our construction, as indicated in (4), is not based on the above formalism. Rather, it uses the formalism of conformal nets. To get acquainted with conformal nets we will start by giving the data of a conformal net and look at an example. In Section 3.3 we will give the complete abstract definition of conformal nets, including the axioms for the above data.

Data. A *conformal net* $\mathcal{A}$ is a monoidal functor from the category[5]

$$\begin{cases} \text{objects: compact, oriented, one-dimensional manifolds with boundary;} \\ \text{morphisms: embeddings that either preserve the orientation on all} \\ \qquad\qquad \text{connected components or reverse the orientation everywhere} \end{cases}$$

to the category

$$\begin{cases} \text{objects: von Neumann algebras;} \\ \text{morphisms: injective homomorphisms and antihomomorphisms.} \end{cases}$$

We require that an embedding $W_1 \hookrightarrow W_2$ is sent to an injective homomorphism $\mathcal{A}(W_1) \hookrightarrow \mathcal{A}(W_2)$ if it preserves orientation, and to an injective antihomomorphism $\mathcal{A}(W_1) \hookrightarrow \mathcal{A}(W_2)^{\mathrm{op}}$ when it reverses orientation.

3.2.1 Conformal nets associated to loop groups

An important class of examples of conformal nets is given by *loop group nets.* Let G be a simply connected compact Lie group equipped with a 'level'. If the group is simple, then a level is just a positive integer $k \in \mathbb{Z}_{\geq 1}$; in general, a *level* is a biinvariant metric on G such that the square lengths of closed geodesics are in $2\mathbb{Z}$. To a one-manifold W we want to assign an algebra. As an intermediate step towards this algebra, we define the group

$$L_W G := \mathrm{Map}_*(W, G) \subset \mathrm{Map}_{C^\infty}(W, G) \tag{5}$$

of all smooth maps $W \longrightarrow G$ that send the boundary ∂W to the unit $e \in G$ and all of whose derivatives are zero at the boundary. Thus, if $W = S$ is a circle, then $L_W G$ is a version of the free loops on G, while if $W = I$ is an interval, it is a version of the based loops on G. The group structure is given by pointwise multiplication in G.

Like the loop group, this group has a central extension by S^1. That central extension is easiest to describe at the level of the Lie algebra $\mathfrak{g}$ of G, where it becomes a central extension by $\mathbb{R}$. The Lie algebra $L_W\mathfrak{g}$ of the loop group $L_W G$

[5] Notice that the source category is not quite monoidal: we cannot take disjoint unions of embeddings that are orientation preserving and embeddings that are orientation reversing.

has a central extension defined by the cocycle[6]

$$c(f,g) = \int_W \langle\, f, \mathrm{d}\, g\,\rangle_k \ , \qquad f,g \in L_W\mathfrak{g}\ . \tag{6}$$

Here, the pairing is given by the metric and depends on the choice of level for G. The corresponding central extension of $L_W G$ is the one that we are after.

The value $\mathcal{A}(W) = \mathcal{A}_{LG,k}(W)$ of the conformal net $\mathcal{A}_{LG,k}$ on the 1-manifold W is then defined as the completion of the group algebra of $L_W G$, with multiplication twisted by the cocycle (6). This is similar to the group algebra of the central extension, but the central S^1 is identified with the S^1 in the scalars. More precisely, we start by forming the free vector space $\mathbb{C}[L_W G]$; since $L_W G$ is a group, this free vector space has the structure of an algebra. The group cocycle $c\colon L_W G \times L_W G \longrightarrow \mathbb{C}^*$ corresponding to (6) allows us to modify the multiplication to $g \cdot_c h \coloneqq c(g,h)\, g\, h$. The associativity is maintained due to the cocycle condition $c(gh,k)\, c(g,h) = c(g,hk)\, c(h,k)$. Finally, the resulting twisted group algebra is not complete, and so we take some completion to make it into a von Neumann algebra.

Loop group nets are made so that they remember all the relevant information about the corresponding loop group. In particular, there is a notion of representation of a conformal net, and the representations of the loop group net agree with the 'positive energy' representation of $L\,G$.[7]

Definition. A *representation* of a conformal net $\mathcal{A}$ is a Hilbert space H equipped with compatible actions of $\mathcal{A}(I)$ for every proper subinterval $I \subsetneq S^1$ of the unit circle.

In Section 4.2.1 we will describe a coordinate-independent approach to the representation theory of conformal nets.

Although S^1 itself has a von Neumann algebra $\mathcal{A}(S^1)$ associated to it, there are examples of conformal nets where $\mathcal{A}(S^1)$ does not act on a representation H. For this reason, one requires actions of the algebras associated to all manifolds $I \subsetneq S^1$ that are *strictly* contained in S. Those must be compatible in the sense that the inclusions $I_1 \hookrightarrow I_2 \subset S^1$ determine the restrictions of the actions. Often, this is equivalent to having a single action of the algebra $\mathcal{A}(S^1)$. We should expect this to hold for loop group nets in particular, although we do not know how this can be proven, except for $G = SU(n)$.

3.2.2 The loop group of $SU(2)$

To get a feeling of what representations of conformal nets are for the case of loop groups, we consider the simplest non-trivial case: the loop group $L\,SU(2)$

[6]Recall that the *central extension* $\hat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is given by the vector space $\mathfrak{g} \oplus \mathbb{C}K$ with bracket

$$[X + \lambda K, Y + \mu K] = [X,Y] + c(X,Y)\, K\ , \qquad X, Y \in \mathfrak{g}\ , \quad \lambda, \mu \in \mathbb{C}.$$

Here the map $c\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ is a *Lie algebra 2-cocycle*: it is antisymmetric and satisfies the cocycle condition $c(X,[Y,Z]) + c(Y,[Z,X]) + c(Z,[X,Y]) = 0$. This ensures that the new bracket is antisymmetric and satisfies the Jacobi identity.

[7]Unfortunately, the fact that representations of $\mathcal{A}_{LG,k}$ are the same as positive energy representation of $L\,G$ is not known in general, even though this is widely expected to be the case. It is known for $G = SU(n)$ due to results of Wassermann [Was98] and partially known for $G = Spin(2n)$ due to Toledano-Laredo [TL97].

of $SU(2)$. Recall that $SU(2)$ has one irreducible representation V_n of dimension $n+1$ for each $n \in \mathbb{N}$. V_0 is the trivial representation, V_1 is the fundamental representation, and so on. For $m \leq n$ the tensor product of two irreducible representations is given by the Clebsch-Gordan decomposition

$$V_m \otimes V_n = V_{n-m} \oplus V_{n-m+2} \oplus \cdots \oplus V_{n+m} \ , \qquad m \leq n \ . \tag{7}$$

This formula is already determined by the simpler relation

$$V_1 \otimes V_n = V_{n-1} \oplus V_{n+1} \ , \qquad n \geq 1 \ .$$

Likewise, the representation theory of the loop group $L\,SU(2)_k$ of $SU(2)$ at level k has irreducible representation $V_0, \cdots, V_k$, where now each of the V_n is an infinite-dimensional Hilbert space. The fusion product of these irreducible representations is determined by

$$V_1 \boxtimes V_n = \begin{cases} V_1 & \text{if } n = 0 \ ; \\ V_{n-1} \oplus V_{n+1} & \text{if } 1 \leq n \leq k-1 \ ; \\ V_{k-1} & \text{if } n = k \ . \end{cases}$$

It is a nice exercise to use the above formulas to find the analogue of (7) for $L\,SU(2)_k$.

We now describe the monoidal structure of $\mathrm{Rep}\,(L\,SU(2)_k)$, the fusion product. This is where conformal nets come in handy. Consider two representations H and K of $\mathcal{A}$, as shown in Figure 13. The two half-circles I and J, with orientations induced by their inclusion in S^1, act as $\mathcal{A}(I) \circlearrowleft H$ and $\mathcal{A}(J) \circlearrowleft K$. Let $\varphi \colon I \longrightarrow J$ be the diffeomorphism that sends the 'north pole' to the 'north pole' and the 'south pole' to the 'south pole' (see again Figure 13). Since φ reverses the orientation, it provides an isomorphism $\mathcal{A}(I) \cong \mathcal{A}(J)^{\mathrm{op}}$ and therefore a right action of $\mathcal{A}(J)$ on H. The fusion product of H and K is then defined to be the Connes fusion $H \boxtimes_{\mathcal{A}(J)} K$. The residual actions of $S^1 \setminus I$ and $S^1 \setminus J$ can then be used to make this product into a new representation of $\mathcal{A}$. Moreover, one can show that, up to isomorphism, the product $H \boxtimes_{\mathcal{A}(J)} K$ is independent of the choice of half-circles I and J.

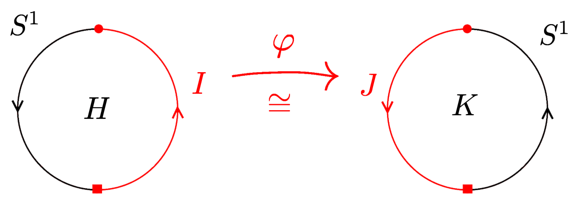


Figure 13: The diffeomorphism used to define the fusion product $H \boxtimes_{\mathcal{A}(J)} K$ between two representations H and K of a conformal net. One can think of H as having actions of all the algebras corresponding to submanifolds of the circle enclosing it, and of K as having actions of algebras living on the other circle.

We repeat that the current state of knowledge about loop group nets is somewhat incomplete, even though it is quite clear what *should* be the case. The cases $G = SU(n)$ is the only one where everything is known.

3.3 Conformal nets revisited

It is convenient to change the source category of conformal nets a bit: henceforth we restrict ourselves to *contractible* compact one-manifolds, i.e., to intervals. The circles can be recovered by gluing tow intervals together.

Definition. A *conformal net*[8] $\mathcal{A}$ is a continuous functor from the category

$$\begin{cases} \text{objects: contractible compact oriented one-manifolds;} \\ \text{morphisms: embeddings} \end{cases}$$

to the category

$$\begin{cases} \text{objects: von Neumann algebras;} \\ \text{morphisms: injective homomorphisms and antihomomorphisms.} \end{cases}$$

An embedding $I \hookrightarrow J$ is sent to a homomorphism $\mathcal{A}(I) \hookrightarrow \mathcal{A}(J)$ if it preserves the orientation, and to an antihomomorphism $\mathcal{A}(I) \hookrightarrow \mathcal{A}(J)^{\mathrm{op}}$ if it reverses the orientation. The hom-sets of the source category carry the C^∞ topology, and there is also a topology on the hom-sets of the target category. It is with respect to these topologies that $\mathcal{A}$, mapping $I \hookrightarrow J$ to an (anti)homomorphism, has to be continuous. Moreover, a conformal net $\mathcal{A}$ is subject to the following axioms:

i) The algebras $\mathcal{A}\big([0,1]\big)$ and $\mathcal{A}\big([1,2]\big)$ commute in, and generate a dense subalgebra of $\mathcal{A}\big([0,2]\big)$;

ii) Denoting the algebraic tensor product by $\otimes_{\mathrm{alg}}$ and the so-called *spatial tensor product* of von Neumann algebras by $\bar{\otimes}$, there exists an extension that makes the diagram commute:

$$\begin{tikzcd} \mathcal{A}\big([0,1]\big) \otimes_{\mathrm{alg}} \mathcal{A}\big([2,3]\big) \arrow[r] \arrow[d] & \mathcal{A}\big([0,3]\big) \\ \mathcal{A}\big([0,1]\big) \bar{\otimes} \mathcal{A}\big([2,3]\big) \arrow[ur, dotted] & \end{tikzcd}$$

iii) The image of the map

$$\big\{\ \varphi \in \mathrm{Diff}\big([0,3]\big)\ :\ \varphi|_{[0,1]} = \mathrm{id}\,,\ \varphi|_{[2,3]} = \mathrm{id}\ \big\} \longrightarrow \mathrm{Aut}\big(\mathcal{A}([0,3])\big)$$

is contained in the set of inner automorphisms of $\mathcal{A}([0,3])$;

iv) There exists a dotted map such that the diagram

$$\begin{tikzcd} \mathcal{A}\big([0,1]\big)^{\mathrm{op}} \otimes_{\mathrm{alg}} \mathcal{A}\big([0,1]\big) \arrow[rr] \arrow[d, "(x\mapsto -x)\otimes(y\mapsto y)"'] & & \mathcal{A}\big([0,2]\big)^{\mathrm{op}} \otimes_{\mathrm{alg}} \mathcal{A}\big([0,2]\big) \arrow[d] \\ \mathcal{A}\big([-1,0]\big) \otimes_{\mathrm{alg}} \mathcal{A}\big([0,1]\big) \arrow[r] & \mathcal{A}\big([-1,1]\big) \arrow[r, dotted] & B\big(L^2\mathcal{A}\big([0,2]\big)\big) \end{tikzcd}$$

commutes.

[8]Note that this definition differs from the definitions in the literature; see e.g. [GF93, KL04a, Lon08]. Our definition is somewhat more general, it allows for more examples.

There is a subtle point that we should mention. Conformal nets have two roles in life. Although the relation with Segal's definition of χCFT may not be clear from the above definition, conformal nets serve as a model for χCFT. On the other hand, they also serve as a model for three-dimensional TQFT, such as Chern-Simons theory. The conformal nets satisfying the above axioms correspond to the TQFTs, while only a subclass of them corresponds to χCFTs. Roughly speaking, the dotted arrow in axiom (iii) has to satisfy a further 'positive energy' condition. Since loop group nets satisfy this condition, they correspond to both a three-dimensional TQFT and a two-dimensional χCFT.

3.4 Frobenius algebra objects

In Section 3.1 we have seen that a χCFT assigns to one-dimensional, compact, oriented smooth manifolds a category $\mathcal{C}$. In our example of interest, this is the category

$$\mathcal{C} := \mathrm{Rep}\,(\mathcal{A}_{LG,k}) \cong \mathrm{Rep}\,(L\,G)$$

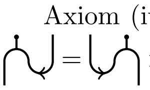
of representations of the loop group net of G at level k. We are interested in objects of $\mathcal{C}$ with a particular kind of extra structure, which can be defined in any monoidal dagger category. Indeed, our category $\mathcal{C}$ consists of Hilbert spaces, so there is a notion of adjoints turning it into a monoidal dagger category with the fusion product.

Definition. A special symmetric *Frobenius algebra object* (we will simply call them Frobenius algebra objects) is an object $Q \in \mathcal{C}$ together with maps

- multiplication $m\colon Q \boxtimes Q \longrightarrow Q$,
- unit $m : 1 \longrightarrow Q$, (here 1 stands for the unit object of $\mathcal{C}$)
- comultiplication $\Delta\colon Q \longrightarrow Q \boxtimes Q$, and
- counit $\varepsilon\colon Q \longrightarrow 1$,

subject to the axioms shown in Figure 14.

Axiom (i) simply states that multiplication and comultiplication are associative and (co)unital. Axiom (ii) is called the *Frobenius condition.* (See Figure 16 for the corresponding axiom for bialgebras.) The third axiom implies that the coalgebra structure on Q is determined by its algebra structure, by taking adjoints.

Axiom (iv) requires Q to be *symmetric*, and is equivalent to the condition  from [FRS02].

Finally, axiom (v) is called the *special* property. The special property means that a Frobenius algebra object is very different from e.g. cohomology rings of manifolds. In particular, it implies that the algebra Q is *semisimple*: any module over Q is semisimple.

The definition of a Frobenius algebra object may look complicated, but a Frobenius algebra is just an algebra satisfying certain properties: everything is determined by the multiplication and unit maps.

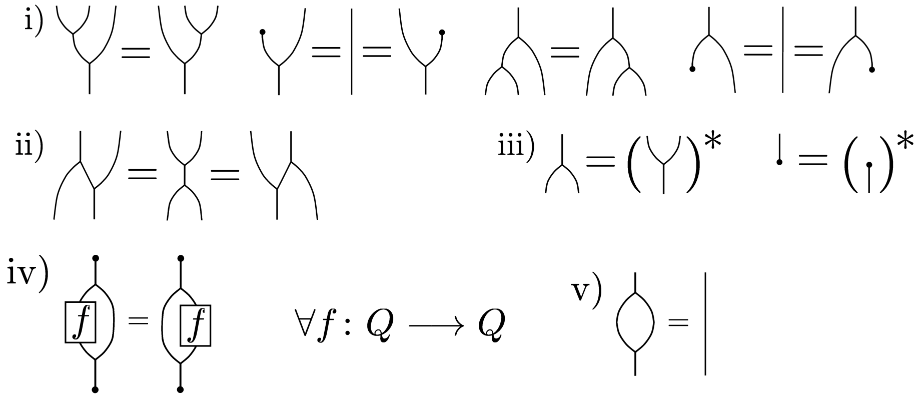


Figure 14: The axioms for a Frobenius algebra object. See Figure 15 for an explanation of string diagrams.

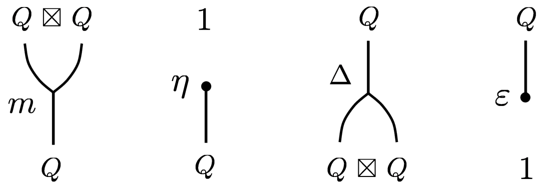


Figure 15: The building blocks of the string diagrams used in the definition of a Frobenius algebra object: (co)multiplication and (co)unit. The diagrams are read from top to bottom. The precise shape of the strings is not important. Because of the distinctive shapes the labels are usually omitted.

3.4.1 Examples

To get a feeling for what Frobenius algebras can look like, let us have a look at some examples. Since every semisimple algebra is a direct sum of simple algebras, we restrict our attention to simple algebras.

A trivial example of a Frobenius algebra object is the unit object of the tensor category.

For another example, consider an object $X \in \mathcal{C}$ and form the Connes fusion $Q = X \boxtimes X^\vee$ of the object with its dual. Then Q is an algebra, indeed a Frobenius algebra. This is the correct generalization of matrix algebras to this context. For instance, taking $\mathcal{C} = \mathrm{Rep}\big(LSU(2)_2\big)$ and $X = V_2$, then $X^\vee \cong V_2$ too and (7) yields $Q = V_0 \oplus V_2$.[9] But $X \boxtimes X^\vee$ is Morita equivalent to the unit object (see Section 3.5.3), so this is still not a very interesting example. However, it leads us to the next example.

Let $\mathcal{C}_k = \mathrm{Rep}\big(LSU(2)_k\big)$ for arbitrary k. In $\mathcal{C}_k$, we have

$$V_0 \boxtimes V_0 = V_0 \ , \quad V_0 \boxtimes V_k = V_k \ , \quad V_k \boxtimes V_0 = V_k \ , \quad V_k \boxtimes V_k = V_0 \ . \tag{8}$$

Thus, we can ask whether, as for $k = 2$, there is an algebra structure on $Q = V_0 \oplus V_k$. It turns out that there are two different monoidal categories that

[9] Another way to understand this example is as follows. The subcategory of $\mathcal{C}$ spanned by V_0 and V_2 is equivalent to $\mathbb{Z}_{/2}$-graded vector spaces as a monoidal category. Inside there, we have the Clifford algebra $\{\, e \mid e^2 = 1, e \text{ is odd} \,\}$.

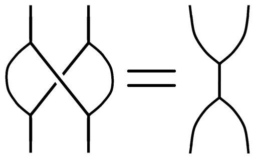


Figure 16: The bialgebra axiom. Bialgebras are also equipped with a product and a coproduct, subject to (co)associativity and (co)unit axioms. Since the bialgebra axiom contains a crossing, bialgebra objects can only be defined in a braided category. On the other hand, the definition of a Frobenius algebra object only requires the ambient category to be monoidal.

satisfy (8): one is the category of $\mathbb{Z}_{/2}$-graded vector spaces, and the other is a version of it where the associator is twisted by a cocycle $c \in H^3(\mathbb{Z}_{/2}, S^1) \cong \mathbb{Z}_{/2}$. The above Q is only an algebra when c is trivial, which happens when k is even.

3.4.2 Classification

There is a beautiful classification of all simple Frobenius algebra objects in $\mathcal{C}_k = \mathrm{Rep}\big(LSU(2)_k\big)$ due to Ostrik [Ost03] (inspired by the CIZ classification of modular invariants for $\hat{\mathfrak{sl}}(2)$ CFTs [CIZ87]) that goes as follows.

Up to Morita equivalence, Frobenius algebra objects in $\mathcal{C}_k$ fall in two infinite families corresponding to the following Dynkin diagrams:

$$\begin{aligned} &\text{type } A_n\text{:} \qquad Q = V_0\ , \quad k = n-1\ , \\ &\text{type } D_n\text{:} \qquad Q = V_0 \oplus V_k\ , \quad k = 2n-4\ . \end{aligned}$$

In addition, there are three exceptional cases:

$$\begin{aligned} &\text{type } E_6\text{:} \qquad Q = V_0 \oplus V_6\ , \quad k = 10\ , \\ &\text{type } E_7\text{:} \qquad Q = V_0 \oplus V_8 \oplus V_{16}\ , \quad k = 16\ , \\ &\text{type } E_8\text{:} \qquad Q = V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}\ , \quad k = 28\ . \end{aligned}$$

For each type we have listed a representative Q of the Morita equivalence class.

3.5 The FRS construction

In Section 3.2.1 we have mentioned that there is a construction that takes a chiral CFT as input, along with a Frobenius algebra object in the category $\mathcal{C}$ provided by the χCFT, and produces a full CFT as output.

In the realm of algebraic quantum field theory, this result is due to Longo and Rehren [LR04, KL04b]. They start with a conformal net and a Frobenius algebra object, and construct a net of von Neumann algebras on $\mathbb{R}^2$ with its Minkowski signature. Such a net assign von Neumann algebras to open subsets of $\mathbb{R}^2$ in such a way that the algebras commute if the opens are causally separated.

Instead of elaborating on this construction we will discuss another approach, which is due to Fuchs, Runkel and Schweigert [FRS02, FRS04a, FRS04b, FRS05, FRS06]. This is a big body of work, and we will only outline some of its aspects.

3.5.1 The partition function

Let us at least describe how to take a χCFT and a Frobenius algebra object and assign a number $Z(\Sigma) \in \mathbb{C}$ to a closed Riemann surface Σ (c.f. the discussion in Section 2.1).

A χCFT assigns to Σ a functor $\mathcal{C}_{\text{in}} \longrightarrow \mathcal{C}_{\text{out}}$. Recall that a non-extended χCFT assigns to a closed surface Σ a linear map $\mathbb{C} \longrightarrow \mathbb{C}$, which is completely determined by an element of $\mathbb{C}$. In the present context, something similar happens. The category Vect is the unit object of the target category LinCat of linear categories. Indeed, LinCat is equipped with a tensor product operation, say $\otimes$, such that, for any linear category $\mathcal{C}$, $\mathsf{Vect} \otimes \mathcal{C} = \mathcal{C}$. Now a linear functor $f\colon \mathsf{Vect} \longrightarrow \mathsf{Vect}$ is completely determined by the image $V := f(\mathbb{C})$, so for any $X \in \mathsf{Vect}$ we have $f(X) = X \otimes V$. The vector space V associated to the functor $\mathcal{C}_{\text{in}} \longrightarrow \mathcal{C}_{\text{out}}$ is called the *space of conformal blocks* associated to Σ by the χCFT. There is also a canonical element $\omega \in V$ provided by the structure of the χCFT: it is the image

$$\mathbb{C} = H_\lambda \longrightarrow H_{f(\lambda)} = V \ , \quad 1 \longmapsto \omega \ ,$$

for $\lambda = \mathbb{C} \in \mathsf{Vect} = \mathcal{C}_{\text{in}}$.

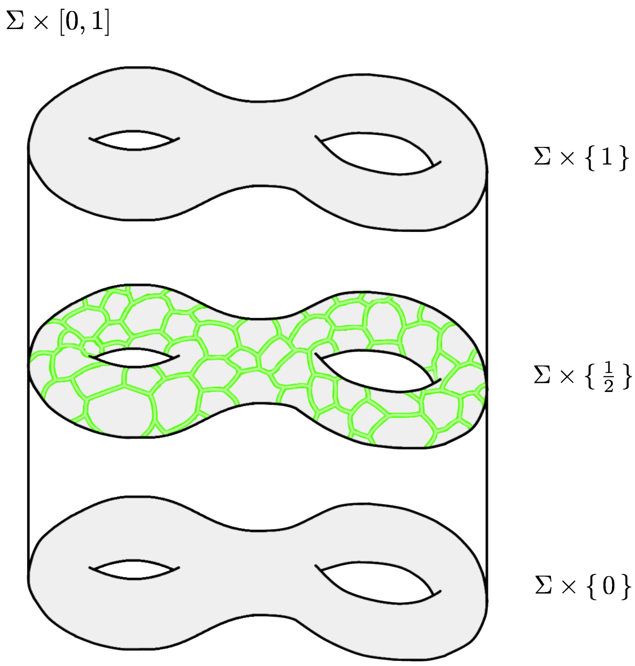


Figure 17: The product of a closed Riemann surface Σ with the unit interval. The middle slice is decorated by a ribbon graph with trivalent vertices. Actually, the middle slice $\Sigma \times \{\frac{1}{2}\}$ should not be there in the picture; it is only needed to explain where the ribbon graph sits.

To see where the Frobenius algebra object comes in, consider the 3-manifold $\Sigma \times [0,1]$ which is obtained by crossing Σ with a unit interval. Decorate the middle slice $\Sigma \times \{\frac{1}{2}\}$ with a *ribbon graph*, whose edges are ‘thickened’ to little

two-dimensional ribbons, as shown in Figure 17. We only allow for trivalent vertices. Now give the ribbons an orientation, so as to get a directed ribbon graph. This can always be done in such a way that each vertex has at least one incoming ribbon and at least one outgoing ribbon. This allows us to further color the graph with the Frobenius algebra object Q and its multiplication m and comultiplication Δ according to the rules shown in Figure 18.

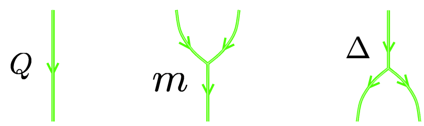


Figure 18: A directed ribbon graph with trivalent vertices can always be colored by these rules (cf. the string diagrams for the (co)multiplication of Frobenius algebra objects in Figure 15).

In order to assign a number to Σ, FRS invoke the existence of a three-dimensional topological quantum field theory (TQFT). The latter assigns to the three-manifold $\Sigma \times [0,1]$ with colored ribbon graph an element $c \in V \otimes \bar{V}$ of the conformal blocks of $\partial\big(\Sigma \times [0,1]\big) = \Sigma \times \bar{\Sigma}$. Here, $\bar{\Sigma}$ denotes the manifold Σ with the reversed orientation. The partition function is then given by

$$Z(\Sigma) = \langle\, c\,,\, \omega \otimes \omega\, \rangle_{V \otimes \bar{V}} \in \mathbb{C}\ .$$

Using the axioms of a (special symmetric) Frobenius algebra object, one can then show that this number does depend neither on the choice of ribbon graph nor on the orientation of the ribbons.

3.5.2 The state space

The next question concerns the state space of the CFT: what is the Hilbert space associated to a circle? For that, one considers the same kind of picture as above, but with a hole, as shown in Figure 19.

We would like to add a single ribbon going down the middle of the hole, in such a way that it is compatible with the ribbon graph on the surface $\Sigma \times \{\frac{1}{2}\}$. Imagine a single strand coming from above, along with another one coming from below. The question is: what do we label these by? Since there are two strands (one from above, and one from below), we are looking for two objects of $\mathcal{C}$. More accurately, we are looking for one object in $\mathcal{C}$ and one in $\bar{\mathcal{C}}$, where the bar now stands for complex conjugation. Equivalently, we are looking for a single object of $\mathcal{C} \times \bar{\mathcal{C}}$. It turns out that this is not quite general enough. What we are really after is an object of $\mathcal{C} \otimes \bar{\mathcal{C}}$, that is, a formal direct sum of objects of $\mathcal{C} \times \bar{\mathcal{C}}$.

To see what the compatibility condition is, consider a juncture of the ribbon in the cylinder with a ribbon from $\Sigma \times \{\frac{1}{2}\}$, as depicted in Figure 20. The ribbon graph can always be arranged so that there is such a junction.

The value (object of $\mathcal{C} \otimes \bar{\mathcal{C}}$) that is assigned to the ribbon on the cylinder has to satisfy the compatibility requirements shown in Figure 21.

It turns out that there is an object that is universal with respect to these

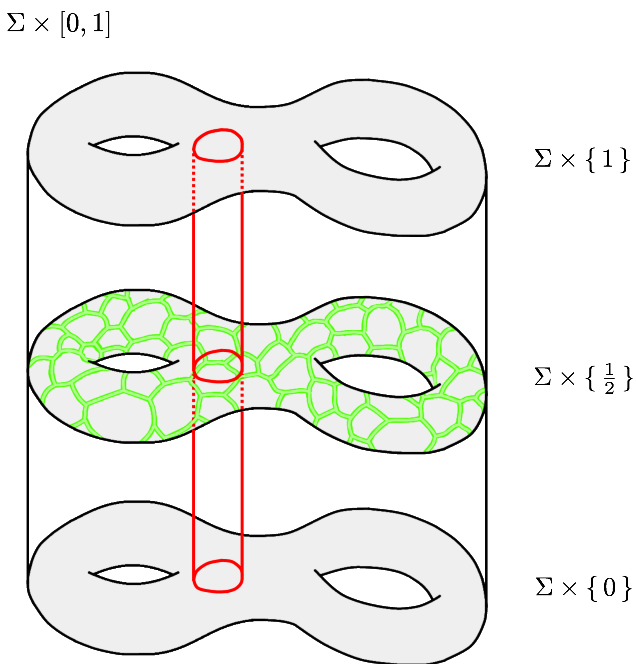


Figure 19: The product of a closed Riemann surface Σ with the unit interval, with a cylinder taken out.

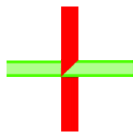


Figure 20: A junction between the ribbon that is dangling down in the cylinder and a ribbon from the graph on $\Sigma \times \{\frac{1}{2}\}$.

properties: the *full centre of* Q, given by[10]

$$Z_{\text{full}}(Q) = \bigoplus_{\mu,\lambda} \mathrm{Hom}_{Q,Q}\big(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}, Q\big) \otimes \lambda \otimes \bar{\mu} \ . \tag{9}$$

Since the Hom-set is a vector space, while $\lambda \in \mathcal{C}$ and $\bar{\mu} \in \bar{\mathcal{C}}$, the full centre of Q is an object of $\mathcal{C} \otimes \bar{\mathcal{C}}$. The $\boxtimes^{\pm}$ in (9) denote the tensor product from above and below (as opposed to from right or from the left), which can be defined because the monoidal category $\mathcal{C}$ is braided. This allows the two Q's to act on the Q in $\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}$ from the left and from the right.

The state space of the full CFT associated to the χCFT together with the Frobenius algebra object Q is then given by

$$H_{\text{full}} \coloneqq \bigoplus_{\mu,\lambda} \mathrm{Hom}_{Q,Q}\big(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}, Q\big) \otimes H_{\lambda} \otimes \overline{H_{\mu}} \ . \tag{10}$$

[10] The 'Z' in (9) stands for centre, and should not be mistaken for the partition function.

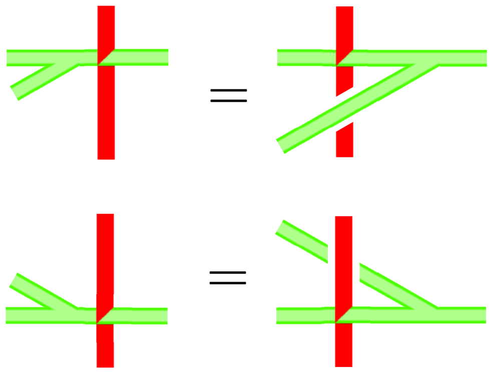


Figure 21: Compatibility requirements between the vertical ribbon in the cylinder (cf. Figure 19) and the ribbons of the graph on $\Sigma \times \{\frac{1}{2}\}$.

In the 'Cardy case', where $Q = 1$ is the unit object, this expression reduces to $H_{\text{full}} = \bigoplus_\lambda H_\lambda \otimes \overline{H_\lambda}$.

Equation (10) is the result of FRS that we were after. The discussion in Section 3.5 mainly serves to provide some motivation for this result, as it is very important for the remainder. Indeed, below we will reproduce this result in the context of extended CFT. We will define what an extended CFT assigns to points and to intervals. Then we will take the two halves of a circle, and fuse them over the algebra associated to their boundary. Comparing the resulting Hilbert space with (10) will provide a check of our formalism.

3.5.3 Defects

We mention one more feature of the FRS construction. Recall that two algebras A and B are said to be *Morita equivalent* if there exist bimodules ${}_AX_B$ and ${}_BY_A$ such that there are isomorphisms

$$ {}_AX \otimes_B Y_A \cong {}_AA_A \quad \text{and} \quad {}_BY \otimes_A X_B \cong {}_BB_B \ . $$

Now if we have two Frobenius algebra objects Q and Q' that are Morita equivalent (with the definition interpreted internally to the category $\mathcal{C}$), the resulting full CFT does not change. In particular, we get the same state space (10).

Kapustin and Saulina [KS11] have a nice way of reinterpeting this fact. Recall the special property, axiom (v) from Section 3.4, of our Frobenius algebra. Figure 22 shows the special property in terms of the directed ribbon graph on $\Sigma \times \{\frac{1}{2}\}$.

In other words: we can fill in the holes in the graph. If we do this everywhere, we get a three-manifold with an embedded surface, and the result looks like in Figure 23.

The three-manifold $\Sigma \times [0, 1]$ now is decorated by a codimension-one *defect* ('surface operator'). According to [KS11] this defect only contains the information of the Morita equivalence class of Q, and not of Q itself. Moreover, one can go back to the ribbons and reinterpret them as actual embedded surfaces whose one-dimensional boundaries are labelled by Q and whose two-dimensional interior corresponds to the defect. Upon filling in the holes in the ribbon graph we

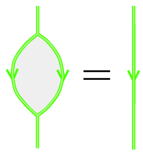


Figure 22: The special property of Frobenius algebra objects allows us to fill in the holes in a ribbon graph.

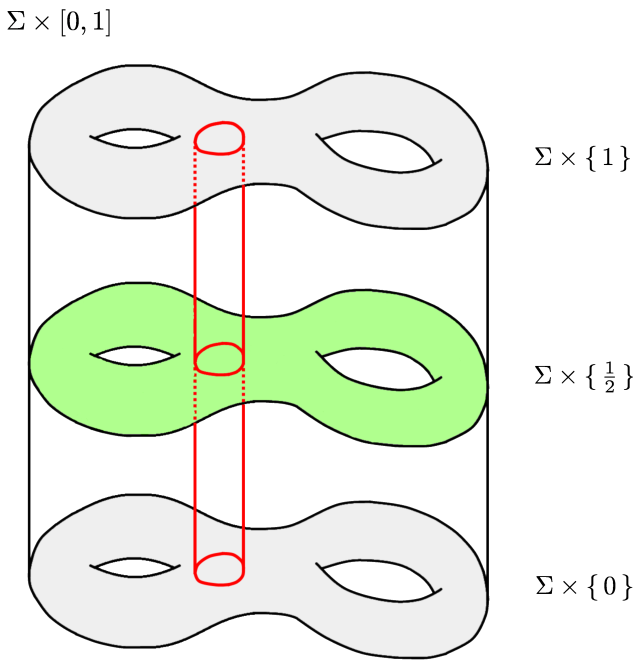


Figure 23: The product of a closed Riemann surface Σ with the unit interval. The holes in the ribbon graph on $\Sigma \times \{\frac{1}{2}\}$ have been filled using the special property of the Frobenius algebra.

get rid of the boundary lines, we no longer see Q, but only its Morita equivalence class, in the form of a defect.

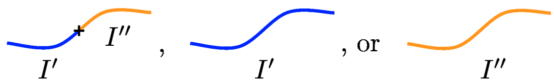
The partition function of Σ is then obtained by evaluating the three-dimensional TQFT on this three-manifold with defect.

3.5.4 Defects between conformal nets

The last ingredient we need in order to make sense of the three-manifold $\Sigma \times [0,1]$ with embedded surface within the formalism of conformal nets is the notion of a defect, leading to defects between conformal nets [BDH09]. For the purpose of the previous discussion, we would only need defects from a conformal net to itself. But in general, defects behave like bimodules: given two conformal nets $\mathcal{A}$ and $\mathcal{B}$, there is a notion of an $\mathcal{A}$-$\mathcal{B}$-defect ${}_{\mathcal{A}}D_{\mathcal{B}}$.

Definitions. A *bicolored interval* is a contractible one-manifold I equipped

with a decomposition $I = I' \cup I''$ that looks like one of



along with a local coordinate at the color-changing point.

A *defect* between conformal nets $\mathcal{A}$ and $\mathcal{B}$ is a functor from the category

$$\begin{cases} \text{objects: bicolored intervals;} \\ \text{morphisms: color preserving embeddings that respect the local coordinate;} \end{cases}$$

to the category

$$\begin{cases} \text{objects: von Neumann algebras;} \\ \text{morphisms: homomorphisms and antihomomorphisms;} \end{cases}$$

sending an embedding $I \hookrightarrow J$ to a homomorphism $D(I) \to D(J)$ if it preserves orientation, and to an antihomomorphism $D(I) \to D(J)^{\mathrm{op}}$ if it reverses orientation. We have that $D(I) = \mathcal{A}(I)$ if I'' is empty, and $D(I) = \mathcal{B}(I)$ if I' is empty. Moreover, D satisfies axioms similar to those of conformal nets.

4 Constructing extended conformal field theory

Until this point we have mostly discussed the work of others. It is time to come back to extended CFT. In this section we will partially construct extended CFT starting from a χCFT that is given to us in the form of a conformal net $\mathcal{A}$, and a Frobenius algebra object $Q \in \mathrm{Rep}(\mathcal{A})$.

Recall from Section 3.2.1 that a representation of $\mathcal{A}$ consists of a Hilbert space H equipped with compatible actions of $\mathcal{A}(I)$ for every $I \subsetneq S^1$. In Section 3.2.2 we have seen how the monoidal structure on $\mathrm{Rep}(\mathcal{A})$ is defined: we identify the left half and the right half of S^1 with $[0,1]$ and set $A := \mathcal{A}\big([0,1]\big)$. This provides a fully faithful embedding of $\mathrm{Rep}(\mathcal{A})$ into the category of A-A-bimodules, and the tensor product on $\mathrm{Rep}(\mathcal{A})$ is inherited from the monoidal structure on A-A-bimodules:

$$(H, K) \longmapsto H \boxtimes_A K\,.$$

We can therefore view the Hilbert space Q as an A-A-bimodule.

4.1 The algebra associated to a point

We start with dimension zero. The algebra that is associated to a point can be defined in the world of A-A-bimodules:

$$B := \mathrm{Hom}\big(L^2A_A, Q_A\big) \tag{11}$$

This is the set of bounded linear maps that commute with the right action of A. The algebra (11) also appears in the work of Longo and Rehren [LR04]; here we present a different construction of it. The reason that this works is the following surprising fact.

Lemma 4.1. *The vector space B is an algebra, and indeed a von Neumann algebra. Moreover, there is an algebra homomorphism $A \longrightarrow B$.*

Proof of Lemma 4.1. Let us sketch the proof. For convenience we abbreviate $1 := L^2 A$ and write $\boxtimes$ instead of $\boxtimes_A$. Recall that the Frobenius algebra object Q comes equipped with a multiplication $m\colon Q \boxtimes Q \longrightarrow Q$, a unit $\eta\colon 1 \longrightarrow A$ and comultiplication $\Delta = m^*$ and $\varepsilon = \eta^*$. We have to define a product, unit, and an involution on B, and show that it is a von Neumann algebra.

Let f and g be elements of B. The *product* of f and g is defined as the composition

$$f \cdot g\colon 1 \xrightarrow{g} Q \cong 1 \boxtimes Q \xrightarrow{f \times 1} Q \boxtimes Q \xrightarrow{m} Q \ .$$

Figure 24 shows how this rule can be represented graphically. Using the diagrams it is clear that the product is associative.

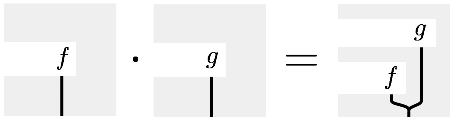


Figure 24: Diagrammatic representation of the product of $f, g \in B$. It should be understood as follows. Consider the diagram on the left, representing f. As with the string diagrams in Section 3.4 we start at the top, which is empty, corresponding to the unit. Then we apply f, which is only linear with respect to the right action of A, so that the left-hand side is 'blocked'. In this way, the diagram exactly shows which operations are allowed algebraically. Finally, the line going to the bottom represents a copy of Q.

The *unit* of B is just the unit map $\eta\colon 1 \longrightarrow Q$ as shown in Figure 25. Together with the above product this determines the algebra structure on B.

Figure 25: Diagrammatic representation of the unit map on B is the same as in the string diagrams for Frobenius algebra objects (cf. Figure 15).

Next, the *involution* is denoted by $\star$ and is defined as the following composition

$$f^\star\colon 1 \xrightarrow{\eta} Q \xrightarrow{\Delta} Q \boxtimes Q \xrightarrow{f^* \times 1} 1 \boxtimes Q \cong Q \ .$$

See Figure 26 for the corresponding diagram.

There is also a map from A to B, sending an element $a \in A$ to the composition of left multiplication by a with the unit:

$$a\colon 1 \xrightarrow{a\cdot} 1 \xrightarrow{\eta} Q \ .$$

This can be represented as shown in Figure 27.

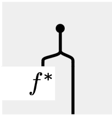


Figure 26: Diagrammatic representation of the involution of $f \in B$.

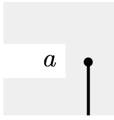
$$A \longrightarrow B\ , \quad a \longmapsto$$ 

Figure 27: Diagrammatic representation of the map $A \longrightarrow B$.

Let $B(H)$ denote the set of bounded operators on the underlying Hilbert space H of Q. The algebra B acts on H via

$$\begin{aligned} &B \longrightarrow B(H)\ , \\ &f \longmapsto \big[\ Q \cong 1 \boxtimes Q \xrightarrow{f \times 1} Q \boxtimes Q \xrightarrow{m} Q\ \big]\ . \end{aligned}$$

The image of $f \in B$ is shown in Figure 28.

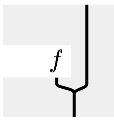


Figure 28: Diagrammatic representation of the left action of B on the Frobenius algebra object Q.

Actually, Q is a B-B-bimodule. The right B-action is shown in Figure 29. It uses the fact that Q is its own dual (the pairing $\varepsilon \circ m$ is nondegenerate) and that for von Neumann bimodules there is a canonical identification between the dual and the complex conjugate. Therefore we can take the complex conjugate $\bar{f}$ of f to get a left A-linear map.

Finally, one can show that the commutant of the left action of B on Q is the right action of B on Q, and vice versa. The algebra B is its own bicommutant, and therefore a von Neumann algebra. $\square$

One can also check that the Hilbert space Q is canonically isomorphic to L^2B as a B-B-bimodule. In order to show that, one has to construct a positive cone $P \subset Q$ (which corresponds to L^2_+B), and define the modular conjugation $J : Q \longrightarrow Q$. Those should then satisfy the axioms listed in [Haa75]. The cone is defined as $P := \{b\,\xi\,b^* \mid b \in B, \xi \in L^2_+A\}$. To construct the modular conjugation, one uses the identification ${}_AQ_A \cong {}_AQ^\vee_A$ coming from the pairing , along with the fact that the dual of a bimodule is always its complex conjugate. We can then define J to be the composite isomorphism $Q \cong Q^\vee \cong \overline{Q}$.

Recall that the zero-manifolds in the source bicategory of our three-tier CFT are generated by two local models: a point with a sign. If B_+ is the von

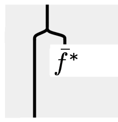


Figure 29: Diagrammatic representation of the left action of B on the Frobenius algebra object Q.

Neumann algebra (11) associated to the point with positive orientation, and B_- the von Neumann algebra associated to the point with negative orientation, then B_+ is canonically isomorphic to B_-^{op}.

One can reinterpret the above construction as that of a defect from $\mathcal{A}$ to $\mathcal{A}$. Namely, there exists a defect D, constructed from the Frobenius algebra object Q, such that $D\big([0,1]\big) = B$. Figure 30 shows the corresponding defect in

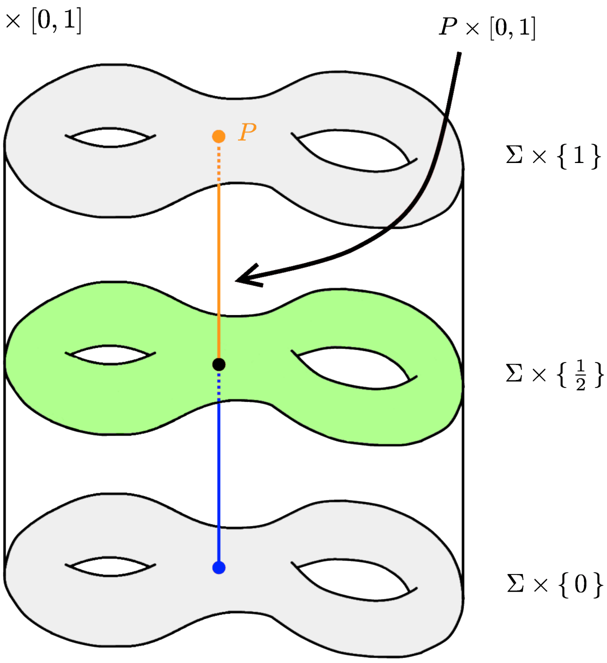


Figure 30: The construction of the algebra B associated to a point $P \in \Sigma$ in the context of the FRS/Kapustin-Saulina construction.

the FRS construction. This is a rather special kind of defect, where the precise location in $[0,1]$ where the colors change is actually not important: the only thing that matters is that the interval $[0,1]$ is genuinely bicolored. Such defects are called *topological defects*. The defect that appeared in Section 3.5.3 is also a topological defect: what the TQFT assigns to a manifold does not change at all when the location of the defect is moved a bit upwards or downwards.

4.2 The bimodule associated to an interval

Points do not have any geometry, and indeed the discussion above was very algebraic. Next we have to decide what to associate to an interval; this will involve some geometry.

We have already seen that, in order to evaluate our extended CFT on a point P, we have to form the product $P \times [0,1]$, and evaluate our defect D on the resulting one-manifold. In the present case we start with an interval I. We are again supposed to cross with $[0,1]$ and do something involving the defect, or, equivalently, with the Frobenius algebra object Q — see Figure 31.

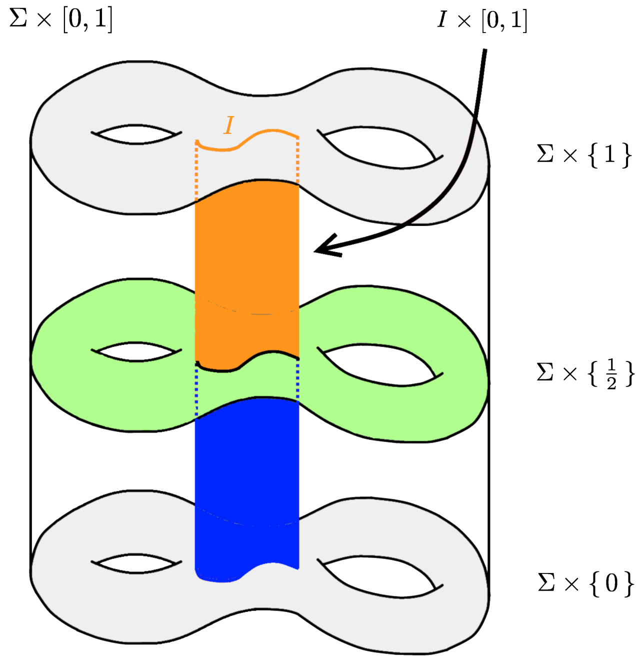


Figure 31: The construction of the Hilbert space $Q(I)$ associated to an interval $I \subset \Sigma$ in the context of the FRS/Kapustin-Saulina construction.

Since we have collars at the ends of our interval, we can smooth out the rectangle $\partial\big(I \times [0,1]\big)$ to a circle. We will see that the extended CFT assigns to I a version of Q modelled on the boundary $\partial\big(I \times [0,1]\big)$: this is a Hilbert space that looks like Q, but which has actions of $\mathcal{A}(J)$ for every $J \subsetneq \partial\big(I \times [0,1]\big)$, as opposed to $J \subsetneq S^1$.

4.2.1 Intermezzo: representations of conformal nets revisited

Before we proceed it is useful to look at a coordinate-independent approach to the representation theory of conformal nets. Consider a circle S: a manifold that is diffeomorphic to S^1, but *without* a choice of such a diffeomorphism. Let $\mathrm{Rep}_S(\mathcal{A})$ denote the category whose objects are Hilbert spaces equipped with compatible actions of $\mathcal{A}(J)$ for every $J \subsetneq S$. $\mathrm{Rep}\,(\mathcal{A})$ is the special case in which S is the unit circle.

Clearly, $\mathrm{Rep}_S(\mathcal{A})$ is equivalent to $\mathrm{Rep}\,(\mathcal{A})$, but there is no canonical way of picking such an equivalence. Also, unlike with $\mathrm{Rep}\,(\mathcal{A})$, there is no canonical monoidal structure on $\mathrm{Rep}_S(\mathcal{A})$. Instead we have an 'external product'. Given three circles S_1, S_2 and S_3 with compatible smooth structures[11] as in Figure 32 there is a canonical map

$$\mathrm{Rep}_{S_1}(\mathcal{A}) \times \mathrm{Rep}_{S_2}(\mathcal{A}) \longrightarrow \mathrm{Rep}_{S_3}(\mathcal{A})\ .$$

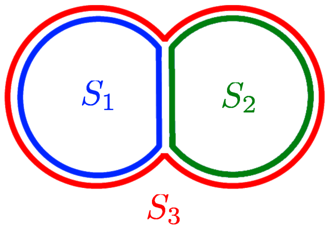


Figure 32: The external product in $\mathrm{Rep}_S(\mathcal{A})$ is defined for each triple of circles with compatible smooth structures.

Now, although there is no canonical equivalence between $\mathrm{Rep}_S(\mathcal{A})$ and $\mathrm{Rep}\,(\mathcal{A})$, we can nevertheless attempt to construct a functor

$$\mathrm{Rep}\,(\mathcal{A}) \longrightarrow\ \mathrm{Rep}_S(\mathcal{A})\ ,$$

and see where we fail. Of course, we could just pick a diffeomorphism $S \longrightarrow S^1$, but that is clearly non-canonical. Let us try the following:

$$H \longmapsto H \underset{\mathrm{Diff}\,(S^1)}{\times} \mathrm{Diff}\,(S^1, S) \qquad (?)\ . \tag{12}$$

The reason why there is an action of $\mathrm{Diff}\,(S^1)$ on H is that whenever a diffeomorphism is supported in a small interval, the corresponding automorphism of $\mathcal{A}(J)$ is *inner*. Thus, there is an element of that algebra associated to the diffeomorphism, which, in turn, acts on H. Those local diffeomorphisms generate $\mathrm{Diff}\,(S^1)$, and so we get an action of $\mathrm{Diff}\,(S^1)$.

The reason that (12) does not quite work is that the choice of algebra elements implementing the given inner automorphism is not unique. Indeed, it is only defined up to phase, and therefore the action of $\mathrm{Diff}\,(S^1)$ on H is only a projective action.

4.2.2 Back to business

We would like to say that the value of the extended full CFT on I is the image of $Q \in \mathrm{Rep}\,(\mathcal{A})$ under the functor

$$\mathrm{Rep}\,(\mathcal{A}) \longrightarrow \mathrm{Rep}_{\,\partial(I\times[0,1])}\,(\mathcal{A})\ .$$

But, as we have seen, at least at first sight, that does not seem to work. The reason that this nevertheless *does* work is that $\partial\big(I \times [0,1]\big)$ has more structure

[11] This is a technical definition that we will not explain here, see our forthcoming paper [BDH13].

than an arbitrary circle S: it has an involution $(x,t) \longmapsto (x,-t)$. Therefore it makes sense to talk about *symmetric diffeomorphisms*, i.e., those diffeomorphisms that commute with the involution.

Thus, for $S = \partial\big(I \times [0,1]\big)$, we can replace (12) by

$$\mathrm{Rep}\,(\mathcal{A}) \longrightarrow \mathrm{Rep}_S(\mathcal{A})\ , \quad H \longmapsto H \underset{\mathrm{Diff}_{\mathrm{sym}}(S^1)}{\times} \mathrm{Diff}_{\mathrm{sym}}(S^1,\, S)\ .$$

Now something very nice happens: the universal central extension of $\mathrm{Diff}\,(S^1)$ splits over $\mathrm{Diff}_{\mathrm{sym}}(S^1)$, and so that group now *does* act on H, and the formula makes sense. Therefore we can define

$$\mathrm{Rep}\,(\mathcal{A}) \longrightarrow \mathrm{Rep}_{\,\partial(I\times[0,1])}\,(\mathcal{A})\ , \quad Q \longmapsto Q(I)\ . \tag{13}$$

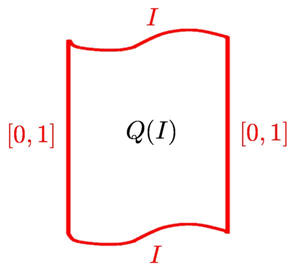


Figure 33: The Hilbert space $Q(I)$ has actions of the copies of B associated to the unit intervals $[0,1]$ on the left and on the right.

To see that $Q(I)$ is indeed a B-B-bimodule, notice that $A = \mathcal{A}\big([0,1]\big)$ has two actions on $Q(I)$, corresponding to the two copies of $[0,1]$ in the boundary of $I \times [0,1]$ (cf. Figure 33). If we identify the boundary $\partial\big(I \times [0,1]\big)$ with the unit circle via some symmetric diffeomorphism that sends the corners on the left to the 'north' and 'south pole' of the circle as illustrated in Figure 34, this identifies the left action of A of $Q(I)$ with the standard left action of A on Q. Now recall from the proof of Lemma 4.1 that the we have an inclusion $A \subset B$ and that the left action of A on Q extends to an action of B on Q in a canonical way. Therefore, the left action of A of $Q(I)$ extends to an action of B.

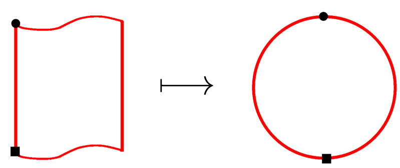


Figure 34: An identification of $\partial\big(I \times [0,1]\big)$ and a standard circle via a symmetric diffeomorphism mapping the corners on the left to the north and south poles of the circle.

Similarly, with the use of a symmetric diffeomorphism as indicated in Figure 35, we can identify the right action of $\mathcal{A}\big([0,1]\big)$ on $Q(I)$ with the standard right action of A on Q, which likewise extends to an action of B.

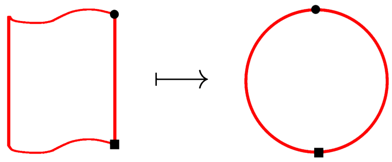


Figure 35: Another identification of $\partial\big(I \times [0,1]\big)$ and a standard circle, now via a symmetric diffeomorphism sending the corners on the right to the poles of the circle.

At this point it is not too difficult to see, using the fact that $Q \cong L^2 B$, that the assignment (13) is compatible with glueing:

$$_B Q(I_1) \boxtimes_B Q(I_2)_B = {}_B Q(I_1 \cup I_2)_B \ .$$

This is of course necessary for our construction to make sense, but it is not very impressive. Let us turn to something more surprising.

4.3 Recovering the state space from the FRS construction

Recall from Section 3.5.2 that the state space (10) of the full CFT from the FRS construction is given by

$$H_{\text{full}} := \bigoplus_{\mu,\lambda} \operatorname{Hom}_{Q,Q}\big(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}, Q\big) \otimes H_{\lambda} \otimes \overline{H_{\mu}} \ .$$

In this section we will show, or at least sketch, how this result can be reproduced with our construction. The idea is to take the unit circle, cut it in half, and fuse the corresponding algebras over $B \mathbin{\bar{\otimes}} B^{\text{op}}$. More precisely, we have the following

Theorem 4.2. *Decompose the unit circle as $S^1 = I \cup J$ such that the intersection $I \cap J$ consists of two points only. Then the fusion of $Q(I)$ with $Q(J)$ over $B \mathbin{\bar{\otimes}} B^{\text{op}}$ (see Figure 36) is canonically isomorphic to H_{full} as a module over the chiral and antichiral algebras.*

For the proof of this theorem we need the following lemma:

Lemma 4.3. *Let Q and B be as in* (11)*, and let H be a module over A. Then H is a Q-module (i.e. we have a map $Q \boxtimes_A H \longrightarrow H$ satisfying the obvious axioms), if and only if it is a B-module extending the action of A (i.e. we have a map $B \otimes H \longrightarrow H$ satisfying the obvious axioms).*

Similarly, a homomorphism $H_1 \to H_2$ is Q-linear iff it is B-linear.

With the help of this lemma, Theorem 4.2 can be proved as sketched — literally — in Figure 37.

4.4 The maps associated to surfaces

Starting from a χCFT and a Frobenius algebra object we have constructed the extended CFT corresponding to zero- and one-dimensional manifolds in the source bicategory. To conclude, we mention what happens to two-dimensional surfaces, and show what the open problem is that has to be solved in order to complete our construction.

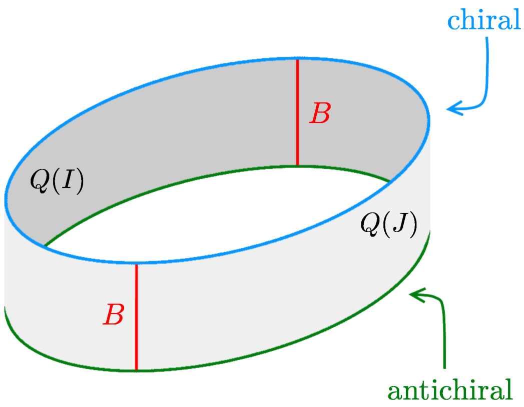


Figure 36: The circle on the top corresponds to the representation of the chiral algebra on H_λ, whilst the circle on the bottom corresponds to the representation of the antichiral algebra on $\overline{H_\mu}$. We have used the action of the two copies of the bigger algebra B (recall that $A \subset B$) to fuse $Q(I)$ with $Q(J)$.

4.4.1 Discs and surfaces with cusps

It is not too hard to see which bimodule map is associated to a disc with conformal structure. We can view the disc as a cobordism from the empty one-manifold to the bounding circle. Thus, we have to construct a map from $\mathbb{C}$ to the Hilbert space H_{full} associated to that circle. This is the same as a choice of vector in that Hilbert space. Moreover, this vector should be invariant under the group $PSL_2(\mathbb{R})$ of Möbius transformations of the circle. The *vacuum vector* in H_{full} is given by $\Omega \otimes \Omega$ in the direct summand $H_0 \otimes H_0$ of H_{full}. Here, Ω is (also) called the vacuum vector in H_0, and H_0 is the *vacuum module* of the conformal net (the unit object in the category $\text{Rep}(\mathcal{A})$). In both cases — i.e. in the case $\Omega \otimes \Omega \in H_{\text{full}}$, and also in the case $\Omega \in H_0$ — the vacuum vector is the unique $PSL_2(\mathbb{R})$-fixed point.

We also have a construction for the bimodule map associated to a surface with two cusps. After a choice of parametrization of the ingoing and outgoing boundaries by the unit interval $[0,1]$, the semigroup $\text{Bigons}\big([0,1]\big)$ of bigons (as in (1)) of the unit interval can be identified with the complexification of the group of those diffeomorphisms of $[0,1]$ that leave a neighbourhood of the endpoints fixed.

By extending the action of $\text{Diff}\big([0,1]\big)$ in a $\mathbb{C}$-linear fashion to the copy of $\text{Bigons}\big([0,1]\big)$ in the chiral sector, and $\mathbb{C}$-antilinearly to the copy of $\text{Bigons}\big([0,1]\big)$ in the antichiral sector, we get the desired actions of $\text{Bigons}\big([0,1]\big)$.

4.4.2 Open problem: ninja stars

The main open problem is the construction of the bimodule map associated to the 'ninja star' depicted in Figure 1 on page 3. We also have to prove a few basic properties of this map, together with one important relation that is shown in Figure 38.

This relation ensures the compatibility between the bimodule map associated

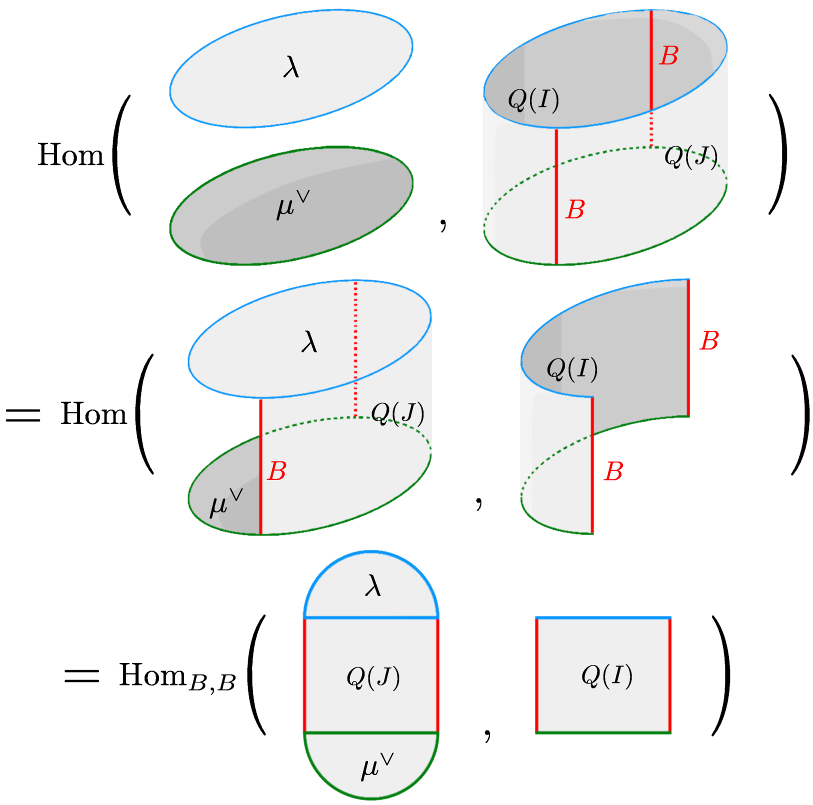


Figure 37: *Proof of Theorem 4.2.* Depict $\mathrm{Hom}_{Q,Q}\big(H_\lambda \otimes \overline{H_\mu}\,,\, Q(I) \boxtimes Q(J)\big)$ as shown on the top. By duality, this is equal to the second line. Now we can flatten the shapes to get to the third line, which corresponds precisely to $\mathrm{Hom}_{Q,Q}\big(\lambda \boxtimes^+ Q \boxtimes^- \mu^\vee, Q\big)$. □

to the ninja star (that we want to construct) and the parts of the extended CFT that we have already constructed. More precisely, it means the following.

Given the 2-morphism from Figure 39a we can form the horizontal composition with the identity 2-morphism on the 1-morphism in Figure 39b to get the result in Figure 39c.

The relation drawn in Figure 39a describes what should happen if we fill in the hole by vertical composition with the disc, viewed as a 2-morphism as indicated in Figure 40. The disc that we fill in corresponds to the lower two-morphism in Figure 41.

Now, any surface can be decomposed into discs and ninja-stars via a simple algorithm: draw closed curves with transverse intersections on the surface, and then replace those intersections by ninja stars (see Figure 42). Given the bimodule map associated to the ninja star and the relation from Figure 38, this decomposition should allow one to construct the full extended CFT from it.

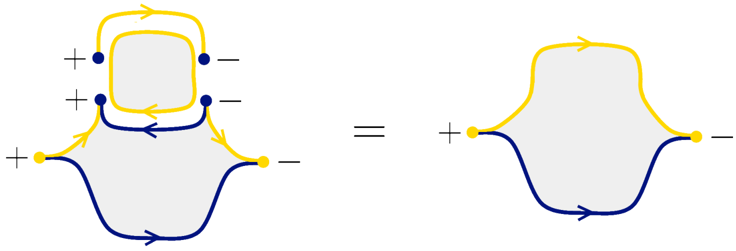
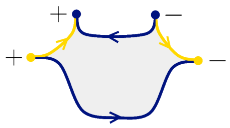
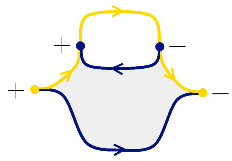


Figure 38: An important relation.



(a) A ninja star.

(b) A one-morphism.



(c) Another 2-morphism.

Figure 39: The 2-morphism on the right is the result of the horizontal compositon of the 2-morphism shown on the left with the identity 2-morphism on the cap in the middle.

Acknowledgements

I am greatly indebted to Stephan Stolz for inviting me to give these lectures, and thus providing the opportunity for this material to get written. I am also very grateful to my student Jules Lamers for compiling a first draft of these notes, and for drawing all the pictures.

References

[Ati89] M. Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. **68** (1989), 175–186.

[BDH09] A. Bartels, C. Douglas, and A. Henriques, *Conformal nets and local field theory*, 2009, arXiv:0912.5307.

[BDH13] A. Bartels, C.L. Douglas, and A. Henriques, *Conformal nets I: coordinate free nets – in preparation*, 2013.

[Bel90] S. Bell, *Mapping problems in complex analysis and the $\overline{\partial}$-problem*, Bull. Amer. Math. Soc. **22** (1990), 233–259.

[CIZ87] A. Cappelli, C. Itzykson, and J.-B. Zuber, *Modular invariant partition functions in two dimensions*, Nuclear Phys. B **280** (1987), no. 3, 445–465.

[Fre93] D.S. Freed, *Extended structures in topological quantum field theory*, Quantum topology (Dayton), 1993, pp. 162–173.

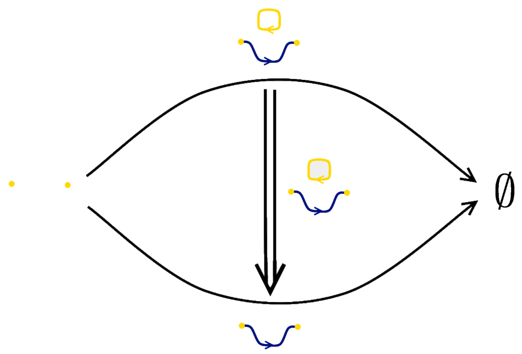


Figure 40: The relation from Figure 38 involves a horizontal composition with the tensor product of the 2-morphism corresponding to the disc and the identity 2-morphism on the (blue) 1-morphism.

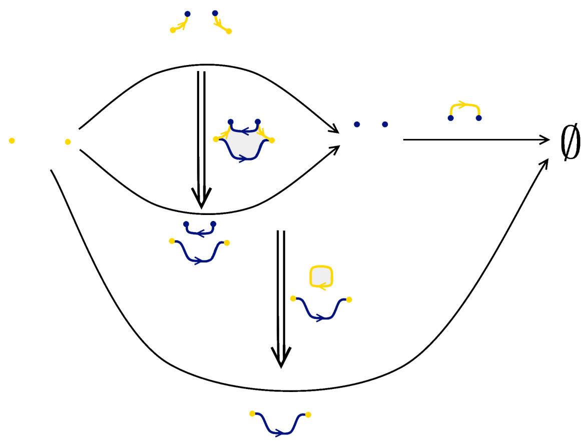


Figure 41: The diagram showing the objects, 1-morphisms and 2-morphism featuring in the important relation from Figure 38.

[FRS02] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators. I. Partition functions*, Nuclear Phys. B **646** (2002), no. 3, 353–497.

[FRS04a] ______, *TFT construction of RCFT correlators. II. Unoriented world sheets*, Nuclear Phys. B **678** (2004), no. 3, 511–637.

[FRS04b] ______, *TFT construction of RCFT correlators. III. Simple currents*, Nuclear Phys. B **694** (2004), no. 3, 277–353.

[FRS05] ______, *TFT construction of RCFT correlators. IV. Structure constants and correlation functions*, Nuclear Phys. B **715** (2005), no. 3, 539–638.

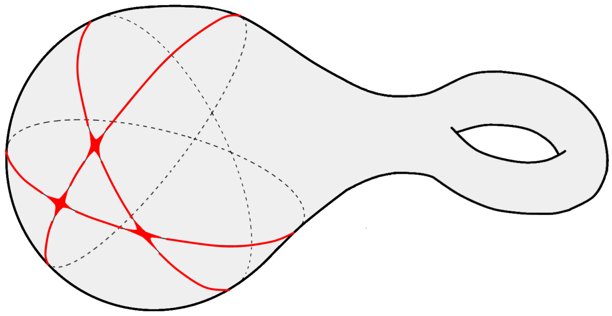


Figure 42: By drawing closed curves on a 2-surface and replacing their junctions by ninja stars, every surface can be decomposed into discs, ninja stars, and intervals.

[FRS06] ———, *TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation*, Theory Appl. Categ. **16** (2006), no. 16, 342–433.

[GF93] F. Gabbiani and J. Fröhlich, *Operator algebras and conformal field theory*, Comm. Math. Phys. **155** (1993), no. 3, 569–640.

[Haa75] U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), no. 2, 271–283. MR 0407615 (53 #11387)

[Hua97] Y.-Z. Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, vol. 148, Birkhuser Boston, Inc., 1997.

[KL04a] Y. Kawahigashi and R. Longo, *Classification of local conformal nets. Case $c < 1$*, Ann. of Math. (2) **160** (2004), no. 2, 493–522. MR 2123931 (2006i:81119)

[KL04b] ———, *Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories*, Comm. Math. Phys. **244** (2004), no. 1, 63–97. MR 2029950 (2005d:81228)

[KS11] A. Kapustin and N. Saulina, *Surface operators in 3d topological field theory and 2d rational conformal field theory*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., 2011, arXiv:1012.0911v1 [hep-th], pp. 175–198.

[Lon08] R. Longo, *Lectures on conformal nets II*, 2008, `http://www.mat.uniroma2.it/~longo/Lecture%20Notes.html`.

[LR04] R. Longo and K.-H. Rehren, *Local fields in boundary conformal QFT*, Rev. Math. Phys. **16** (2004), no. 7, 909–960.

[Lur09] Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. MR 2555928 (2010k:57064)

[Ost03] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups **8** (2003), no. 2, 177–206.

[Pos03] H. Posthuma, *Quantization of Hamiltonian loop group actions*, Ph.D. thesis, 2003.

[RS06] D. Radnell and E. Schippers, *Quasisymmetric sewing in rigged Teichmüller space*, Commun. Contemp. Math. **8** (2006), no. 4, 481–534.

[Seg] G. Segal, *Sewing Riemann surfaces together*, (unpublished preprint).

[Seg88] ______, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., 1988, pp. 165–171.

[Seg04] ______, *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser., vol. 308, ch. The definition of conformal field theory, pp. 421–577, Cambridge Univ. Press, 2004.

[Seg07] ______, *Elliptic cohomology*, London Math. Soc. Lecture Note Ser., vol. 342, ch. What is an elliptic object?, pp. 306–317, Cambridge Univ. Press, 2007.

[ST04] S. Stolz and P. Teichner, *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser., vol. 308, ch. What is an elliptic object?, pp. 247–343, Cambridge Univ. Press, 2004.

[TL97] V. Toledano Laredo, *Fusion of Positive Energy Representations of $LSpin(2n)$*, Ph.D. thesis, University of Cambridge, 1997, `arXiv:math/0409044` [math.OA].

[Was98] A. Wassermann, *Operator algebras and conformal field theory. III. Fusion of positive energy representations of $LSU(N)$ using bounded operators*, Invent. Math **133** (1998), no. 3, 467–538.

[Zhu96] Y. Zhu, *Modular Invariance of Characters of Vertex Operator Algebras*, J. Amer. Math. Soc. **9** (1996), 237–302.